Unconditional Mean, Volatility and the Fourier - Garch Representation*

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Abstract

This paper proposes a new model called Fourier-Garch that is a modification of the popular Garch(1,1). This modification allows for time-varying first and second moments via means of Flexible Fourier transforms. A nice feature of this model is its ability to capture both short and long run dynamics in the volatility of the data, requiring only that the proper frequencies of the Fourier transform be specified. Several simulations show the ability of the Fourier series to approximate breaks of an unknown form, irrespective of the date and/or location of breaks. It is shown that the main cause of the long run memory effect seen in stock returns is the result of a time varying first moment. Finally, it is shown that allowing only the second moment to vary over time is not sufficient to capture the high persistence observed in lagged returns.

Keywords: Garch, Structural Change, Unconditional Volatility
JEL-Classification: G12, G29

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1 Introduction

Recently there has been an upsurge in interest in modeling the nonstationarities present in the volatility of financial data. The clustering and the persistence of volatility of asset returns has well been documented. The IGARCH model of Engle and Bollerslev (1986) for instance, describes in a parsimonious way the high persistence in the conditional volatility of stock returns while the underlying process remains strictly stationary. Alternatively, Granger (1980) and Granger and Joyeux (1980) model the long memory or the long range dependence of a series of log-returns as a fractionally integrated process to allow the autocorrelation function to decay very slowly, in a fashion characteristic of stock returns. However, seminal papers from Granger and Joyeux (1986) and Lamoureux and Lastrapes (1990) and more recently from Diebold and Inoue (2001), Mikosch and Starica (2004), Starica and Granger (2005), and Perron and Qu (2004) argue that the high persistence close to unit root and long memory both in the first and second moments may actually be caused by structural changes in the level or slope of an otherwise locally stationary process for the second moment. Diebold and Inoue (2001) argue that this is due to switching regimes in the data. Mikosch and Starica (2004) provide theoretical evidence that changes in the unconditional mean or variance induce the statistical tools (sample ACF, periodogram) to behave the same way they would if used on stationary long-range dependent sequences. Starica and Granger (2005) also deliver evidence against global stationarity. Finally, Perron and Qu (2004) conclude that the S&P 500 return series is best described as a stationary short memory process contaminated by mean shifts.

These remarkable results imply that a good model for volatility should take into account the possibility of a time varying unconditional second-moment and possibly, of a time varying first moment as well. A first such attempt is the Spline-Garch of Engle and Rangel (2005) who model long-run volatility non-parametrically using an exponential quadratic spline. However, they do so only for the second moment. Further, Starica and Granger (2005) use step functions to approximate nonstationary data locally by stationary models. They apply their methodology to the S&P 500 series of returns covering a period of seventy years of market activity and find that most of the dynamics are concentrated in shifts of the unconditional variance.

However, these models pose several problems. While spline functions may lead to over fitting, step functions may not give smooth approximations. Even major breaks, such as the stock market crash of 1929 and the oil price shocks of the 1970s, did not display their full impacts immediately. Structural changes may thus take longer to extinguish, which suggests they should be modeled as smooth or gradual changing processes. These arguments motivate the present paper to propose a new approach to model the long-run first and second moments as smooth processes. I call the new model Fourier-Garch because it uses the Flexible Fourier transform of Gallant (1981) (i.e., an expansion of a periodic function in terms of an infinite sum of sines and cosines). The model can be extended to incorporate long-run volatility in the mean model. Flexible Fourier transforms have been used in the literature to approximate nonlinear structures in several ways. For instance, Becker et al. (2001) uses Fourier transforms to model inflation and money demand as having smooth changes in the intercept. Also, Enders and Lee (2006) and Becker et al. (2006) propose new unit root and stationarity tests that use the Fourier approximation to model the unknown shape of the structural breaks in macro time series. The main advantage is that the issue of estimating the shape and location of the breaks is reduced to the selection of the proper frequency of the Fourier sine and cosine terms. I show below how Fourier transforms can be used to approximate various types of breaks.

I apply the new model to several of the largest stocks from the S&P 500 to estimate volatility persistence in stock returns. Based on the discussion above, I consider several competing models. The basic Fourier-Garch representation keeps the first moment constant, while it allows the second moment to change smoothly over time. One extension to the basic model allows both the first and second moment to vary over time, while a second extension incorporates the long run volatility in the model for the mean. For each model I check the sum of the estimated coefficients in the equation for conditional volatility, which is indicative of the so called long-memory effect. The results show that allowing only the second moment to vary over time does not significantly reduce the persistence effect. In fact, the difference between this model and the simple Garch(1,1) is negligible. However, the extended model that allows the first moment to vary over time as well reduces the persistence effect by more than half of the value suggested by Garch(1,1). The evidence suggests that the persistence effect seen in stock returns is mainly a result of the mispecification of the model for the mean.

The paper is structured as follows: section 2 discusses in more detail the performance of the Fourier series to approximate various types of structural breaks; section 3 introduces the basic Fourier-Garch model and its extensions; section 4 discusses the empirical estimates of the long memory effect using four different models; section 5 concludes.

2 Nonlinear Trend Approximation with Fourier Transforms

The general approach to account for breaks is to approximate them using dummy variables. However, this approach has several undesirable consequences. First, one has to know the exact number and location of the breaks. These are not usually known and therefore need to be estimated. This in turn introduces an undesirable pre-selection bias (see Maddala and Kim (1998)). Second, use of dummies suggests sharp and sudden changes in the trend or level. However, for low frequency data it is more likely that structural changes take the form of large swings in the data which cannot be captured well using only dummies. Breaks should therefore be approximated as smooth processes (see Leybourne et al. (1998) and Kapetianos et al. (2003)).

Flexible Fourier transforms, originally introduced by Gallant (1981), are able to capture the essential characteristics of one or more structural breaks only by using a small number of low frequency components. This is true because a break tends to shift the spectral density function towards frequency zero. Below is illustrated the ability of Fourier transforms to capture nonlinear trends.

Using a simple form for the mean model, one can allow the intercept μ_t to be a deterministic function of time:

$$y_t = \mu_t + \gamma t + \varepsilon_t \tag{1}$$

where the drift term is written as:

$$\mu_t = c_0 + \sum_{k=1}^{s} c_k \sin\left(2\Pi kt/T\right) + \sum_{k=1}^{s} d_k \cos\left(2\Pi kt/T\right); s \leq T/2 \tag{2}$$

In the above formulation ε_t is a stationary disturbance term with variance σ_ε^2, s is the maximum number of frequencies, k is a particular frequency, and T is the total number of observations. The drift term represents the Fourier approximation written as a deterministic function of sine and cosine terms. Note that by imposing $\alpha_k = \beta_k = 0$ one gets the constant mean or trend return specification. In contrast to other possible series expansions (e.g. Taylor series for instance) the Fourier expansion has the advantage of acting as a global approximation (see Gallant (1981)). This property is obtained even if one specifies a small number of frequencies. In fact, Enders and Lee (2006) argue that a large value of s in a regression framework uses a lot of the degrees of freedom and leads to an over-fitting problem.

To illustrate the approximation properties of a Fourier series, I consider first a single frequency in the Data Generating Process (DGP):

$$\mu_t = c_0 + c_k \sin\left(2\Pi k t/T\right) + d_k \cos\left(2\Pi k t/T\right); \tag{3}$$

where k is the single frequency selected in the approximation, and c_k and d_k represent the magnitudes of the sinusoidal terms.

I generate several possible patterns for the occurrence of a break. Thus, for $T = 500$, I simulate one break, two breaks, and trend breaks both in the middle and towards the extremes. The Appendix illustrates cases for temporary, permanent, and reinforcing breaks in panels 1 through 9. As in Enders and Lee (2006), Panels 1 and 2 display approximations for breaks towards the end of a series. In panel 3, the series has a temporary, though long-lasting break. Panels 4 and 5 display permanent breaks in opposite directions, while in Panel 6 the breaks are in the same direction. Finally, Panels 7-9 depict breaks in the intercept and slope of a trending series. I estimate the coefficients of the sinusoidal terms by performing a simple regression of y_t on μ_t and a time trend.

One can draw several conclusions based on the visual inspection of the graphs. First, a single frequency $k = 1$, or two cumulative frequencies $n = 2$, can approximate a large variety of breaks. Second, the Fourier transform approximates well even when the breaks are asymmetric (see Panels 1 and 2). Third, a Fourier series works best when the break is smooth over time, which means it may not be suited for abrupt and sharp breaks of short duration (see Panel 5). An additional frequency of $k = 2$ can improve the fit in this situation. Interested readers are referred to Enders and Lee (2006) and Becker et al. (2006) who have a longer discussion on the properties of the Fourier approximations. I introduce next a new model to approximate long-run volatility.

3 A New Model for Unconditional Volatility

As the introductory part suggested, the simple Garch(1,1) setup for estimating the conditional volatility may not be appropriate because it implies a long-run level of the volatility which is constant. However, the extensive work done so far regarding the presence of various shifts in stock returns suggests that structural changes in the second moment induce global nonstationarity. This invalidates the use of the simple Garch(1,1). It is known that breaks shift the spectral density function towards frequency zero. This indicates that the frequencies used are towards the low end of the spectrum (see Enders and Lee (2006)). A simple visual inspection of the autocorrelation function and periodogram of absolute returns of S&P 500 confirms this fact:

[Insert Figure 1]

As you can note from the top graph, the most important frequencies that impact the absolute returns are at the low end of the sample spectrum; this is an indication of structural breaks. Both graphs confirm the presence of long memory in financial returns - slow decay with lags still significant at the 200th lag. These findings suggest the use of the following model whose aim is to capture various unknown shifts in long-run volatility. I call it the basic Fourier-Garch:

$$r_t = \mu + v_t \sqrt{u_t h_t}, \ \ where \ \ v_t | I_{t-1} \sim iid(0,1) \tag{4}$$

$$h_t = (1 - \alpha - \beta) + \alpha \left(\frac{(r_{t-1} - \mu)^2}{u_{t-1}} \right) + \beta h_{t-1} \tag{5}$$

$$u_t = exp \left[a_0 + \sum_{k=1}^{s} \left(a_k \sin \frac{2\pi k t}{T} + b_k \cos \frac{2\pi k t}{T} \right) \right] ; s \leq T/2 \tag{6}$$

The model preserves the parsimony of the Garch(1,1) model while it allows the unconditional expectation of the volatility to be a function of time and of cycles of different frequencies. A simple extension allows the unconditional mean to be a deterministic function as well - higher unconditional variance certainly requires higher unconditional mean. The time varying first moment is also approximated using a Fourier representation:

$$\mu_t = c_0 + \sum_{k=1}^{s} \left(c_k \sin \frac{2\pi kt}{T} + d_k \cos \frac{2\pi kt}{T} \right) \tag{7}$$

Given its flexible setup, the Fourier-Garch captures both short and long-run dynamics. Note that:

$$E(r_t - \mu)^2 = E(v_t^2 u_t h_t) = u_t E(h_t) = u_t \tag{8}$$

I use an exponential representation of the Fourier transform to ensure its positivity. Goodness of fit measures like the BIC and AIC criterions are employed to detect the proper number of frequencies. They are given by:

$$AIC = -lnL + 2n, L = -\sum_{t=1}^{T} \left[ln(h_t u_t) + \frac{(r-\mu)^2}{h_t u_t} \right] \tag{9}$$

$$BIC = -lnL + nln(T), L = -\sum_{t=1}^{T} \left[ln(h_t u_t) + \frac{(r-\mu)^2}{h_t u_t} \right] \tag{10}$$

Here n denotes the number of parameters estimated in the model. The advantage of using the AIC and BIC criterions (as opposed to minimizing the sum of squared residuals, for instance) is that they include a penalty for the additional estimated parameters. Throughout the estimation, the criterions employ only integer frequencies.

I stress the advantage of using a time varying first moment using a sample of forty years of daily data of S&P 500 absolute returns:

[Insert Figure 2]

Note the better fit of the second model which augments the basic Fourier-Garch representation with a time varying intercept as in equation (7). However, given the presumption that a higher long run volatility requires a higher long-run return, I also employ the Fourier-M model that includes the unconditional time-varying volatility in the equation for the mean:

$$r_t = \gamma u_t + v_t \sqrt{u_t h_t}, \;\; where \;\; v_t | I_{t-1} \sim iid(0, 1) \tag{11}$$

In this way, both the first and the second moment change over time while the underlying model ensures a parsimonious representation.

One way to asses the persistence or long memory in stock returns is to compute the sum of the slope coefficients in conditional volatility. If the sum is close to one, then conditional volatility is said to be almost integrated and it displays very slow time decay. However, the support for long-memory weakens if one finds that a changing first and/or second moment is responsible for the persistence effect. If the sum of the coefficients is significantly less than one after one accounts for shifts in the unconditional mean or volatility, then one can conclude that volatility is a stationary process that suffers from structural shifts (Perron and Qu (2004)).

A sample of daily returns on the S&P500 from 1963:01:02 to 2005:2:30 illustrates this discussion. The best representation specifies a single frequency both for the mean and for the unconditional volatility:

[Insert Figure 3]

Note the slow and gradual increase of long run volatility from the 1960s until the 1980s. Also note that the estimated long run volatility of the 1990s is lower than the one for previous decades, which is consistent with market facts.

4 Model Validation and Persistence Effects

I use several representative S&P 500 stocks to assess the long memory effect of stock returns using the new models. I choose the first thirteen stocks of the index according to their market percentage participation as of March 2005. Table 1 shows their ticker, sector classification, and percent of total assets. The data has been obtained from the Center of Research in Security Prices made available through the WRDS database. The longest sample period available is 1926:01:02 - 2005:12:30 and corresponds to Exxon, IBM, Chevron, Phillip-Morris, and General Electric. Other stock returns have shorter sample periods (i.e., Procter & Gamble from 1929:01:02 onwards, Pfizer and Johnson & Johnson start in 1944; Bank of America and Intel from 1972, while the rest start in 1986). For each stock return, I choose exogenously an integer or cumulative frequencies from 1 to 5 for which I compute the AIC and BIC criterions. According to Enders and Lee (2006), a value greater than 5 uses a lot of the degrees of freedom and leads to an over-fitting problem.

The AIC and BIC criterions indicate that in most cases the best representation is the basic Fourier-Garch(1,1) model[1]. This is not surprising given that the BIC criterion favors more parsimonious representations. The Appendix shows the full set of results for each stock under analysis. Several exceptions to the findings above are noteworthy. In the case of Microsoft for instance, both criterions select the Fourier-M model to be the optimal representation. Also, the Fourier-M model gives the best fit for Chevron as well. Note that the basic Fourier-Garch(1,1) and the Fourier-M models have very close values for the BIC and SBC criterions. This is true because they estimate the same number of parameters (i.e.

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[1] The coefficients of the sine and cosine terms with up to 5 frequencies are significant at the 5% level both for the basic and for the extended models. However, given that in the model for the mean each additional frequency requires the estimation of two more coefficients, the additional penalty increases the values of the AIC and BIC criterions relative to the ones for the basic model.

six coefficients). For the rest of the returns, the increased penalty due to the additional coefficients that are estimated in the models with two or more cumulative frequencies is greater than the better fit that is obtained. Therefore, the single frequency representation fits the data best for all models.

Next I check whether the selected returns display the long memory property that is usually observed in financial data. To this end, I estimate four competing models:

- the common Garch(1,1) developed by Bollerslev (1987) denoted M_0 ;
- the basic Fourier-Garch(1,1) with constant first moment, denoted M_1 ;
- the Fourier-Garch(1,1) with a time varying first moment, denoted M_2 ;
- the Fourier-M(1,1) with long-run volatility in the mean, denoted M_3 .

Clearly, model M_2 provides the best reduction of the persistence effect for most series. For eleven of the thirteen stock returns considered, the long memory effect is dramatically reduced in many instances by half or more (i.e. GE, Pfizer, IBM, Phillips-Morris, Chevron, Intel, Procter & Gamble, Exxon, Johnson & Johnson, Citigroup and Bank of America). Note that the basic representation (i.e., model M_1) has little impact on overall persistence in the short-run volatility. In most cases, its persistence is only slightly lower than the one of the Garch(1,1) representation. This is surprising given that this model, according the the AIC and BIC criterions, gives the best fit in eleven of the thirteen stocks considered. Note that model M_3 clearly outperforms model M_1 in terms of reduced long memory effect as well. The main conclusion is that allowing only for the second moment to vary over time is not enough to account for the strong persistence effect observed in financial returns. However, in contrast to the basic model, a time-varying first moment in the equation for the mean reduces significantly the persistence in short run volatility.

5 Conclusion

This paper proposes a new model to estimate the short and long run dynamics in financial data that takes into account the possibility of a time varying first and second moment. The Flexible Fourier transform of Gallant (1981) approximates the unknown date and shape of any structural break in the first and second moment as a smooth process. This paper shows that Fourier series are able to approximate a wide variety of breaks of unknown forms. The basic Fourier-Garch representation modifies the popular Garch(1,1) to include a time varying unconditional variance. I propose two extensions to the basic model. The first extension additionally specifies a time varying first moment, while the second extension includes the long-run volatility in the equation for the mean. The results suggest that persistence still remains significant in the short run volatility of the basic model. However, the so called long memory effect disappears if one includes a time varying first moment in the model for the mean. This suggests that conditional volatility persistence is an artifact of the mispecification of the model for the mean.

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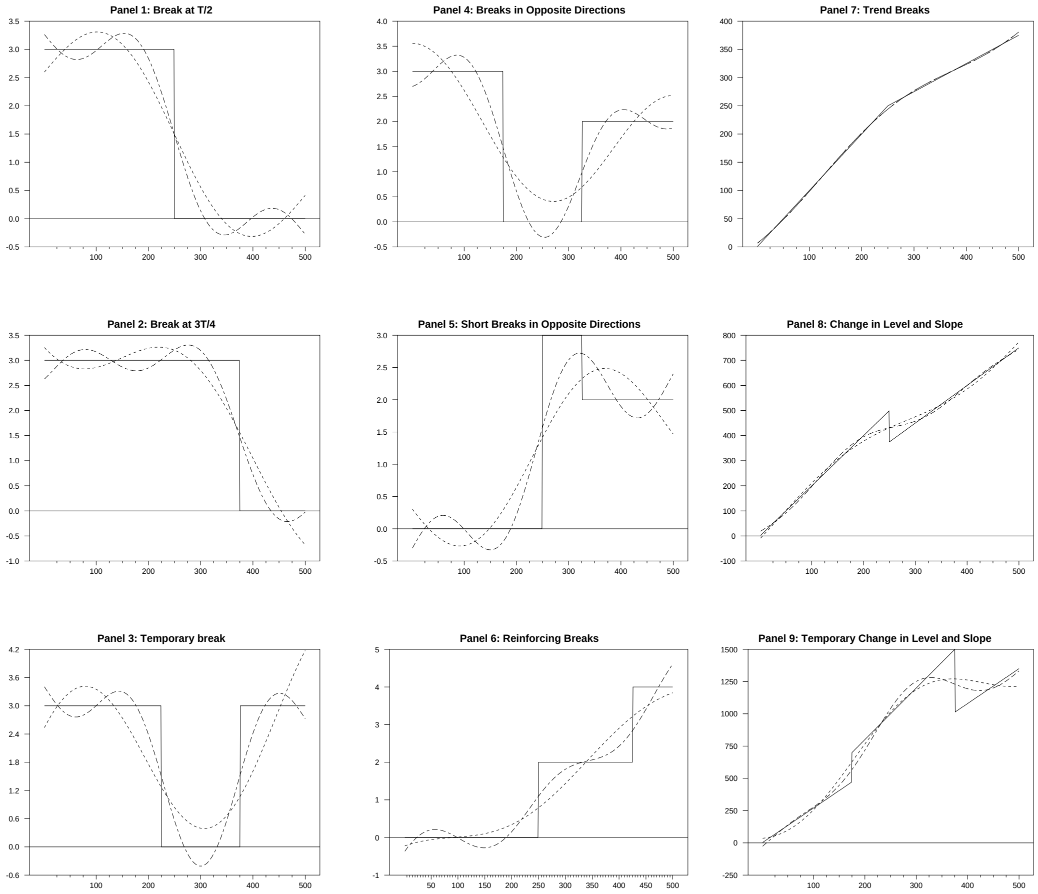
Appendix

Table 1: Market Capitalization of 13 companies on S&P 500 as of 2/28/2006

Ticker	Issue Name	Sector	% of Total Assets
XOM	Exxon Mobil Corp	Energy	3.19
GE	General Electric Co.	Industrials	3
MSFT	Microsoft Corp.	Industrials	2.12
C	Citigroup Inc	Financials	2.03
BAC	Bank of America Corp.	Financials	1.84
PG	Procter & Gamble	Consumer Staples	1.73
PFE	Pfizer Inc.	Health Care	1.67
AIG	American Intl. Group Inc.	Financials	1.49
JNJ	Johnson & Johnson	Health Care	1.48
MO	Altria Group Inc.	Consumer Staples	1.29
CVX	Chevron Corp New	Energy	1.09
IBM	International Business Mach.	Information technology	1.09
INTC	Intel Corp	Information Technology	1.07

Approximation of Structural Breaks with Fourier Transforms



Series: — ; One frequency: - - ; Two cumulative frequencies: -- -

SP 500 Graphs

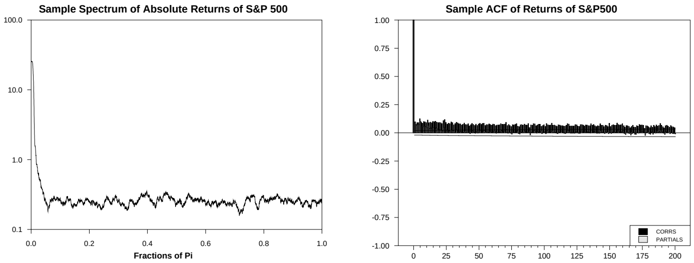


Figure 1: Left - Sample Spectrum of absolute returns of S&P 500; Right - Sample ACF returns of S&P 500

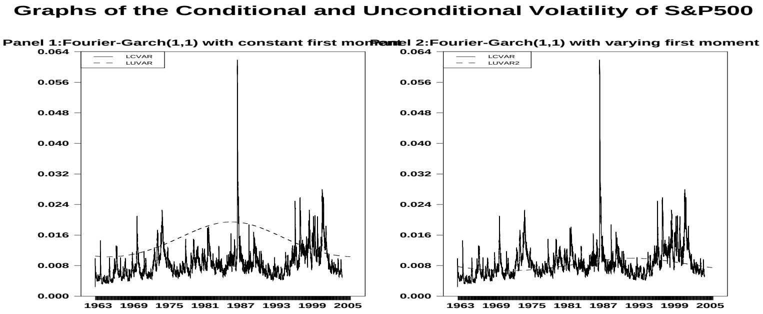


Figure 2: LCVAR = conditional volatility; LUVAR = unconditional volatility

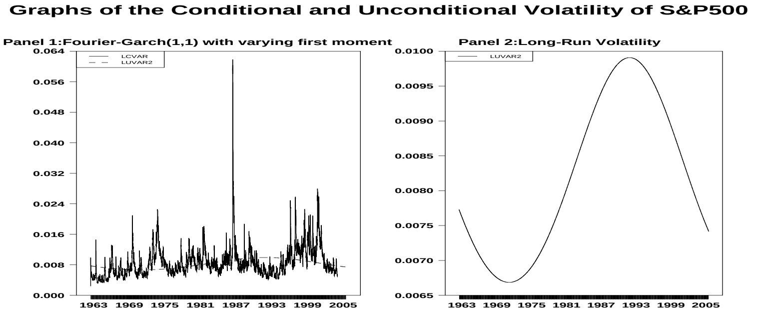


Figure 3: Unconditional volatility with varying first moment

Microsoft

Table 2: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	1.53153	40.63109	10.46847
2	5.62881	57.76155	10.37119
3	9.61467	74.55801	10.38533
4	13.50727	91.70639	10.49273
5	17.53896	108.77126	10.46104
1 (with mean shifts)	5.69638	57.82912	10.30362
1 (Fourier-M)	**1.49355**	**40.59311**	**10.50645**

Table 3: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	**0.06820**
M_1: Fourier-Garch(1,1) with constant mean	0.10247
M_2: Fourier-Garch(1,1) with time-varying mean	0.22565
M_3: Fourier-M(1,1) with long-run volatility in the mean model	0.09550

MSFT:Conditional and Unconditional Volatility from Fourier-Garch(1,1)

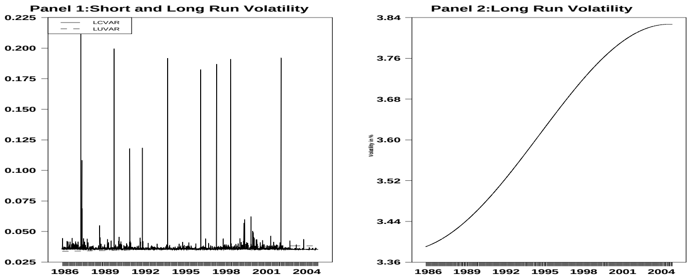


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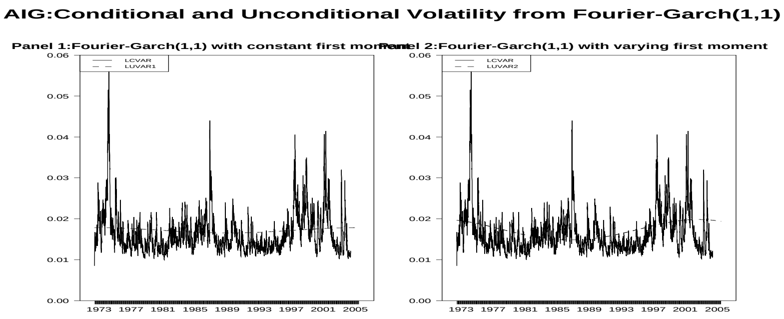
Table 4: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	**0.98896**	**40.2862**	**11.01104**
2	4.98715	61.21770	11.01285
3	8.98678	79.27496	11.01322
4	12.98666	97.33248	11.01334
5	16.98661	115.39007	11.01339
1 (with mean shifts)	5.13206	61.36261	10.86794
1 (Fourier-M)	1.00155	43.17446	10.9845

Table 5: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	0.98024
M_1: Fourier-Garch(1,1) with constant mean	0.98034
M_2: Fourier-Garch(1,1) with time-varying mean	0.97753
M_3: Fourier-M(1,1) with long-run volatility in the mean model	**0.96798**



General Electric

Table 6: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	**0.04772**	**47.72355**	**11.95228**
2	4.07575	67.64353	11.92425
3	8.04628	87.50600	11.95372
4	12.04656	107.39823	11.95344
5	16.04628	127.28989	11.95372
1 (with mean shifts)	4.37495	67.94311	11.62505
1 (Fourier-M)	0.07400	47.74983	11.92600

Table 7: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	0.99256
M_1: Fourier-Garch(1,1) with constant mean	0.99013
M_2: Fourier-Garch(1,1) with time-varying mean	**0.80713**
M_3: Fourier-M(1,1) with long-run volatility in the mean model	0.99261

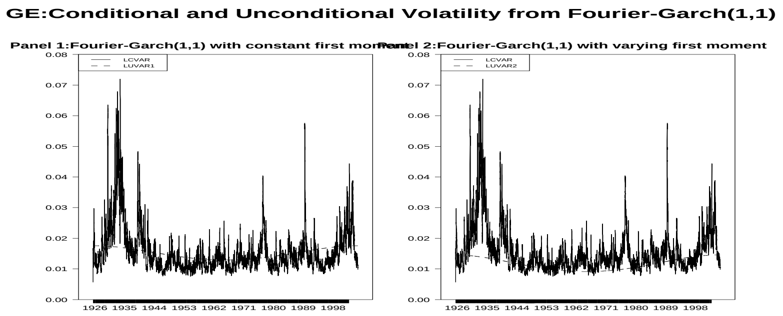


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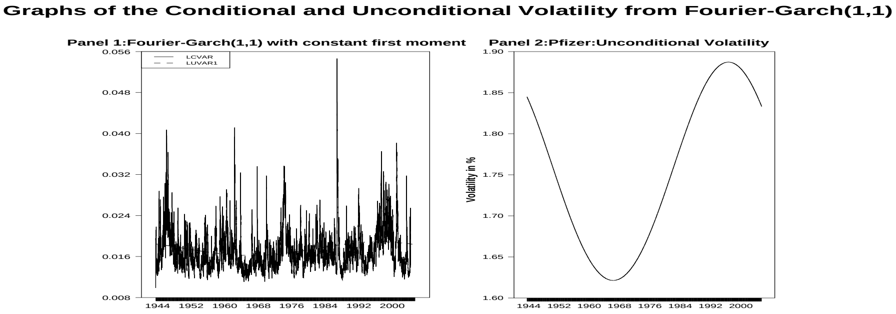
Table 8: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	**0.36115**	**46.39615**	**11.63885**
2	4.37894	65.75895	11.62106
3	8.36153	85.08654	11.63847
4	12.36133	104.43134	11.63867
5	16.36044	123.77545	11.63867
1 (with mean shifts)	6.80988	68.18989	9.19012
1 (Fourier-M)	0.36447	46.39947	11.6353

Table 9: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	0.97707
M_1: Fourier-Garch(1,1) with constant mean	0.90421
M_2: Fourier-Garch(1,1) with time-varying mean	**0.40066**
M_3: Fourier-M(1,1) with long-run volatility in the mean model	0.85240



Graphs of the Conditional and Unconditional Volatility from Fourier-Garch(1,1)

Panel 1:Fourier-Garch(1,1) with constant first moment

Panel 2:Pfizer:Unconditional Volatility

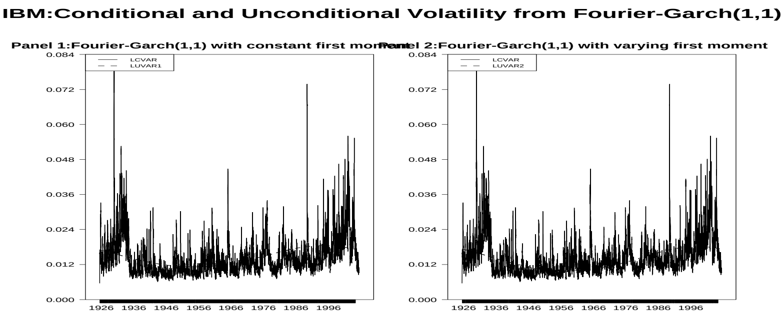
IBM

Table 10: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	**0.02809**	**47.70382**	**11.97191**
2	4.02617	65.59433	11.97383
3	8.02657	87.48677	11.97343
4	12.02611	107.37778	11.97389
5	16.02618	127.26979	11.97382
1 (with mean shifts)	4.93026	68.49804	11.06974
1 (Fourier-M)	0.02874	47.70457	11.97126

Table 11: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	0.99180
M_1: Fourier-Garch(1,1) with constant mean	0.96291
M_2: Fourier-Garch(1,1) with time-varying mean	**0.51090**
M_3: Fourier-M(1,1) with long-run volatility in the mean model	0.95930



IBM:Conditional and Unconditional Volatility from Fourier-Garch(1,1)

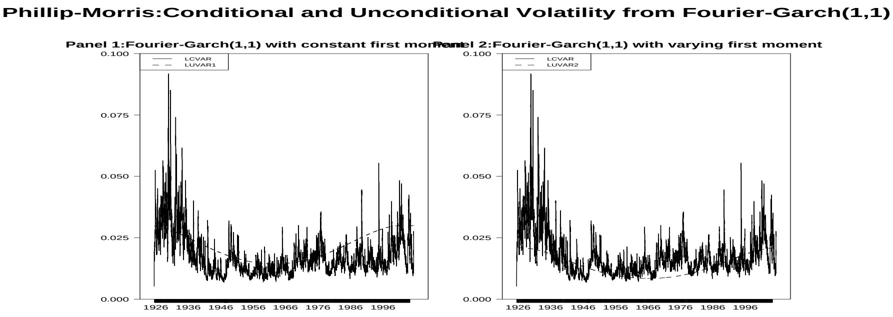
Phillip-Morris

Table 12: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	**0.07408**	**47.74991**	**11.92592**
2	4.07656	67.64472	11.92344
3	8.07646	87.53666	11.92354
4	12.07659	107.42826	11.92341
5	16.07650	127.32011	11.92350
1 (with mean shifts)	4.61120	68.17898	11.38880
1 (Fourier-M)	0.07915	47.75498	11.92085

Table 13: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	0.99877
M_1: Fourier-Garch(1,1) with constant mean	0.99251
M_2: Fourier-Garch(1,1) with time-varying mean	**0.75108**
M_3: Fourier-M(1,1) with long-run volatility in the mean model	0.98887


Phillip-Morris:Conditional and Unconditional Volatility from Fourier-Garch(1,1)

Panel 1:Fourier-Garch(1,1) with constant first moment

Panel 2:Fourier-Garch(1,1) with varying first moment

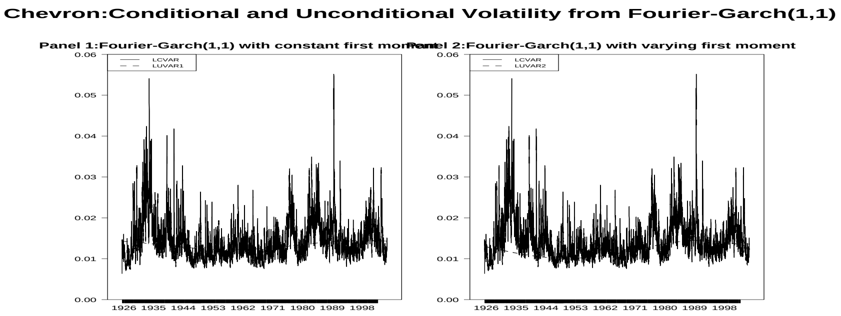
Chevron

Table 14: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	0.04402	47.71985	11.95598
2	4.03508	67.60324	11.96492
3	8.03493	87.49513	11.96507
4	12.03498	107.38665	11.96502
5	16.03475	127.27836	11.96525
1 (with mean shifts)	4.64509	68.21287	11.35491
1 (Fourier-M)	**0.03891**	**47.71474**	**11.96109**

Table 15: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	0.98704
M_1: Fourier-Garch(1,1) with constant mean	0.98373
M_2: Fourier-Garch(1,1) with time-varying mean	**0.75108**
M_3: Fourier-M(1,1) with long-run volatility in the mean model	0.96577



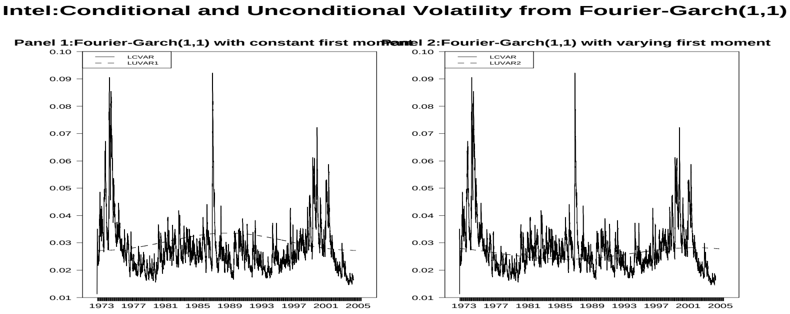
Intel

Table 16: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	**1.14228**	**48.81811**	**10.85772**
2	5.15282	61.38337	10.84718
3	9.15283	79.44101	10.84717
4	13.15209	97.49791	10.84791
5	17.15037	115.55383	10.84963
1 (with mean shifts)	5.36649	61.59704	10.63351
1 (Fourier-M)	1.16193	43.33484	10.83807

Table 17: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	0.99185
M_1: Fourier-Garch(1,1) with constant mean	0.99206
M_2: Fourier-Garch(1,1) with time-varying mean	**0.64818**
M_3: Fourier-M(1,1) with long-run volatility in the mean model	0.98175



Intel:Conditional and Unconditional Volatility from Fourier-Garch(1,1)

Panel 1:Fourier-Garch(1,1) with constant first moment

Panel 2:Fourier-Garch(1,1) with varying first moment

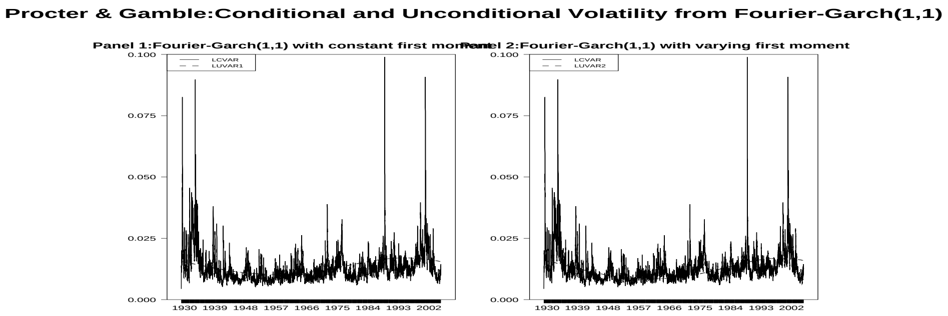
Procter & Gamble

Table 18: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	**0.04260**	**47.71843**	**11.95740**
2	4.04091	67.23956	11.95909
3	8.04095	87.03926	11.95905
4	12.04081	106.83878	11.95919
5	16.04101	126.63864	11.95899
1 (with mean shifts)	4.96195	68.16060	11.03805
1 (Fourier-M)	0.04738	47.44637	11.95262

Table 19: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	0.99595
M_1: Fourier-Garch(1,1) with constant mean	0.96786
M_2: Fourier-Garch(1,1) with time-varying mean	**0.29293**
M_3: Fourier-M(1,1) with long-run volatility in the mean model	0.96698



Procter & Gamble:Conditional and Unconditional Volatility from Fourier-Garch(1,1)

Panel 1:Fourier-Garch(1,1) with constant first moment

Panel 2:Fourier-Garch(1,1) with varying first moment

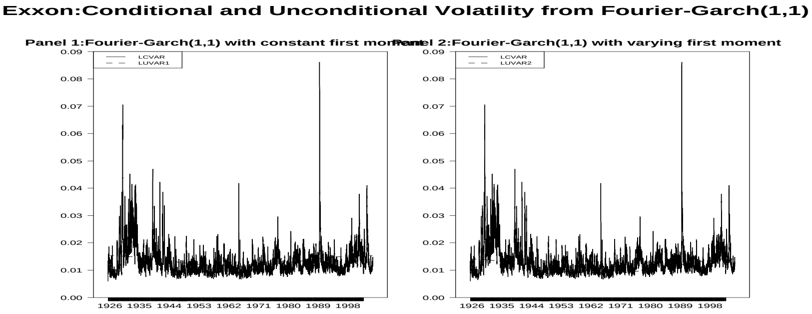
Exxon

Table 20: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	**0.00297**	**47.67880**	**11.99703**
2	4.00538	67.57354	11.99462
3	8.00512	87.46532	11.99488
4	12.00527	107.35694	11.99473
5	16.00523	127.24884	11.99477
1 (with mean shifts)	5.48779	69.05557	10.51221
1 (Fourier-M)	0.00764	47.68347	11.99236

Table 21: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	0.98333
M_1: Fourier-Garch(1,1) with constant mean	0.95727
M_2: Fourier-Garch(1,1) with time-varying mean	**0.54307**
M_3: Fourier-M(1,1) with long-run volatility in the mean model	0.95894



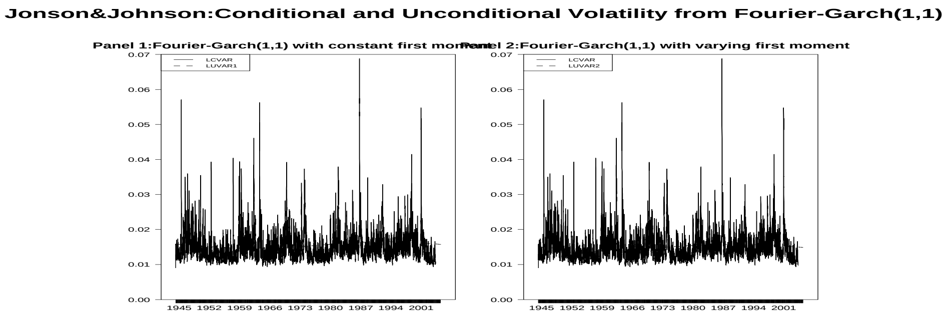
Johnson & Johnson

Table 22: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	**0.33368**	**46.28993**	**11.66632**
2	4.33369	65.60870	11.66631
3	8.33364	84.92740	11.66636
4	12.33365	104.24616	11.66635
5	16.33326	123.56452	11.66674
1 (with mean shifts)	5.25722	66.53223	10.74278
1 (Fourier-M)	0.33508	46.29133	11.66492

Table 23: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	0.95222
M_1: Fourier-Garch(1,1) with constant mean	0.88250
M_2: Fourier-Garch(1,1) with time-varying mean	**0.01595**
M_3: Fourier-M(1,1) with long-run volatility in the mean model	0.90133



Bank of America

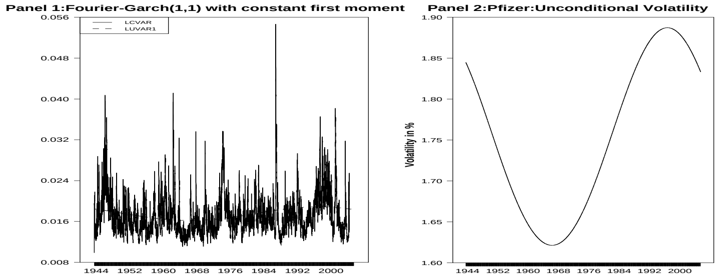
Table 24: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	**0.89500**	**43.06791**	**11.10500**
1 (Fourier-M)	0.91235	43.08526	11.08765

Table 25: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	0.72790
M_1: Fourier-Garch(1,1) with constant mean	0.82855
M_2: Fourier-Garch(1,1) with time-varying mean	**0.15189**
M_3: Fourier-M(1,1) with long-run volatility in the mean model	0.90798

Graphs of the Conditional and Unconditional Volatility from Fourier-Garch(1,1)



Panel 1:Fourier-Garch(1,1) with constant first moment

Panel 2:Pfizer:Unconditional Volatility

Citigroup

Table 26: AIC, BIC and Log-Likelihood

Frequencies	AIC	BIC	Log-Likelihood
1	**1.60934**	**40.51240**	**10.39066**
2	5.63425	57.50500	10.38592
3	9.61749	74.45592	10.38251
4	13.61364	91.41976	10.38636
5	17.61543	108.38923	10.38457
1 (with mean shifts)	5.93613	57.80688	10.06387
1 (Fourier-M)	4.75741	43.66047	7.24259

Table 27: Persistence in financial volatility

Model Representation	Sum of α and β coefficients
M_0: Garch(1,1)	1.00104
M_1: Fourier-Garch(1,1) with constant mean	0.90388
M_2: Fourier-Garch(1,1) with time-varying mean	**0.57667**
M_3: Fourier-M(1,1) with long-run volatility in the mean model	0.99360