FI520 Project Report

User Guide

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A Guide to Valuing First-To-Default Options using

Copulas

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Abstract

The purpose of the program is to price a first-to-default option written on a basket of two names (assets), rated AA and BBB for instance. Under the contract, the counterparty selling protection pays $1,000,000 if one of the names defaults over a T-year period. Copula functions are employed to model and price the first-to-default option.

Chapter 1: **Introduction**

This is a study case of a credit-risk product in the sense that the derivative contract contains the so-called counterparty risk - the counterparty in the contract may default by the time it has to honor its obligations. To be more specific, assume that an entrepreneur is funding a project whose value is $V(T)$ with debt issued in the form of a zero coupon bond with a face value $\overline{DD}$. The debt is reimbursed at date T . Assuming that the asset side of the firm $V(T)$ follows a GBM, under the risk neutral-measure we have:

$$dV(t) = rV(t)dt + \sigma_V V(T)d\tilde{z}_t \tag{1.1}$$

Then, by using the standard Black-Scholes model the value of equity (i.e. what is left after bondholders have been repaid) is:

$$C(t) = V(t)N(d_1) - exp(-r(T-t))\overline{DD}N(d_2) \tag{1.2}$$

$$d_1 = \frac{ln(V(t)/\overline{DD}) + (r + \sigma_V^2/2)(T-t)}{\sigma_V \sqrt{T-t}} \tag{1.3}$$

$$d_2 = d_1 - \sigma_V \sqrt{T-t} \tag{1.4}$$

The leverage amount is given by:

$$d = \frac{exp(-r(T-t))\overline{DD}}{V(t)} \tag{1.5}$$

The main difference with respect to the Black-Scholes framework is that in this case not only the volatility of the underlying asset σ_V, but also its current value $V(t)$ cannot be observed on the market. What we observe instead is the value of equity $C(t)$. To close the model, from Ito's lemma we have:

$$\sigma_C = \sigma_V N(d_1)\frac{V(t)}{C(t)} \tag{1.6}$$

The default probability, under the risk neutral measure is recovered as $N(-d_2)$. Our aim in this project is thus to find the joint default probability of two such assets.

The main issue here is how to model the dependence between the two assets since we have to deal with the bivariate distribution (relationship) of the two assets. Thus the problem becomes very complicated by the fact that we would have to deal with a double integration problem. This is where Copulas come in very handy.

Chapter 2: **Copulas**

2.1 What are Copulas?

Having a multivariate distribution, the univariate marginals and the multivariate dependence structure can be separated and the multivariate structure can be represented by a copula. For instance, in the bivariate case if we let $X \sim F$, $Y \sim G$ and $(X, Y) \sim H$ the copula is defined as:

$$H(x, y) = C(F(x), G(y)) \tag{2.1}$$

Using the marginals we get that:

$$h(x, y) = f(x) \times g(y) \times c(F(x), G(y)) \tag{2.2}$$

The above representation shows the advantages of using copulas. Specifically, the copulas enable us to tackle the problem of the specification of the marginal univariate distributions separately from the specification of the market comovement and dependence. Moreover, it allows us to get away from the concept of linear dependence which has been shown to be a weak concept of dependence since the latter includes many other types of dependencies all possibly captured by the copulas:

- positive and negative dependence
- exchangeable (flexible) dependence
- dependence decreasing with lag if theres a time index
- singularities on some curves or surfaces

2.2 Some Copulas

A simple natural extension of the Black-Scholes model to a bivariate setting consists in assuming the returns from the two projects are jointly normally distributed. Li(2000) proposes the Gaussian copula to capture the joint default dependency of a portfolio formed by two assets:

$$C^{Ga} = \int_{-\infty}^{-d_{2A}} \int_{-\infty}^{-d_{2B}} \frac{1}{2\pi\sqrt{1-\rho^2}} exp\left(\frac{2\rho sw - s^2 - w^2}{2(1-\rho^2)}\right) dsdw \tag{2.3}$$

The Gaussian copula assumes that the dependency between the two assets is symmetric. However, other representations, like the one-parameter Archimedean Copulas treat the comovement of the assets as being asymmetric over time. Some examples are given below:

Gumbel Copula:

$$C(v, z) = exp\left(-\left[(-lnv)^{\alpha} + (-lnz)^{\alpha}\right]^{1/\alpha}\right), \alpha \geq 1; \tag{2.4}$$

Clayton Copula:

$$C(v,z) = max\left[\left(v^{-\alpha} + z^{-\alpha} - 1\right)^{-1/\alpha}, 0\right], \alpha \in [-1,0) \cup (0,+\infty) \tag{2.5}$$

Frank Copula:

$$C(v,z) = \frac{-1}{\alpha} ln\left[1 + \frac{(exp(-\alpha v) - 1)(exp(-\alpha z) - 1) - 1}{exp(-\alpha) - 1}\right], \alpha \in (-\infty,0) \cup (0,+\infty) \tag{2.6}$$

This class above of copulas is particularly suited at capturing the Positive Quadrant Dependency: $X \sim F$, $Y \sim G$ in our case are more likely to be large together or to be small together.

Chapter 3: **Credit-linked Products**

3.1 A Credit Derivative Example

The majority of copula functions have been devoted to credit risk and products whose payoff depends on the performance of a basket of obligations from several names. A first-to-default swap is a credit derivative where the counterparty offering protection pays a sum (i.e., a fixed amount) at the first event of default out of a basket of credit exposures.

Our first-to-default derivative pays one dollar if at least one of the two credit exposures defaults by time T. The risk neutral probability of paying the protection is equal to the probability of the event that both names will survive beyond time T. Formally,

$$FTD = exp\left[-r(T-t)\right]\left[1 - Pr(\tau_1 > T, \tau_2 > T | \Im_t)\right] \tag{3.1}$$

where FTD denotes *first-to-default* and $\tau_i, i = 1, 2$ denote the default times of the two names. The price of the credit risk derivative in terms of copula functions can be written as:

$$FTD = exp\left[-r(T-t)\right]\left[1 - \overline{C}(\overline{Q}_1(T), \overline{Q}_2(T) | \Im_t\right] \tag{3.2}$$

where $Q_i(T), i = 1, 2$ denote the marginal default probabilities and $\overline{Q}_i(T), i = 1, 2$ denote the marginal survival probabilities. Using the duality relationship between a copula function and its survival copula we obtain

$$FTD = exp\left[-r(T-t)\right]\left[Q_1(T) + Q_2(T) - C(Q_1(T), Q_2(T) | \Im_t\right] \tag{3.3}$$

where the price depends separately on the marginal default probabilities and the joint default probability. The choice for the exact copula functions depends on one's own perspectives and assumptions about the market co-movements of the assets. Therefore, if the co-movement appears asymmetric (i.e., tail dependence) one can employ the one-parameter Archimedean copulas. However, if the joint default risk appears rather symmetric, one can opt for the multivariate normal distribution and use the Gaussian copula to represent dependence:

$$C(Q_1(T), Q_2(T) | \Im_T) = N\left[N^{-1}(Q_1(T)), N^{-1}(Q_2(T)), \rho\right] \tag{3.4}$$

The case of perfect positive dependence with highest possible default risk is given by

$$FTD_{max} = exp\left[-r(T-t)\right]\left[max(Q_1(T), Q_2(T)) | \Im_t\right] \tag{3.5}$$

and the opposite case of independence is given by

$$FTD_{\perp} = exp\left[-r(T-t)\right]\left[Q_1(T) + \overline{Q}_1(T) Q_2(T) | \Im_t\right] \tag{3.6}$$

So, as the value of the copula function increases with dependence, the value of the first-to-default product decreases.

3.2 Program Inputs/Outputs

The program takes as inputs the equity prices of the assets, the par values of debt, the risk free interest rate, volatilities and the times to maturity. It also requires the specification of α and the correlation coefficient ρ.

The output gives the leverage amount of the two names, the marginal default probabilities, the copula values and the prices of the first-to-default derivative for each copula representation assumed.

3.3 Conclusions

Copulas are very useful to model credit risk and price credit derivatives. They are fairly straightforward to implement (especially the one-parameter Archimedean family of copulas). They augment the Black-Scholes framework to account for non-linear relationships among assets.

3.4 Appendices

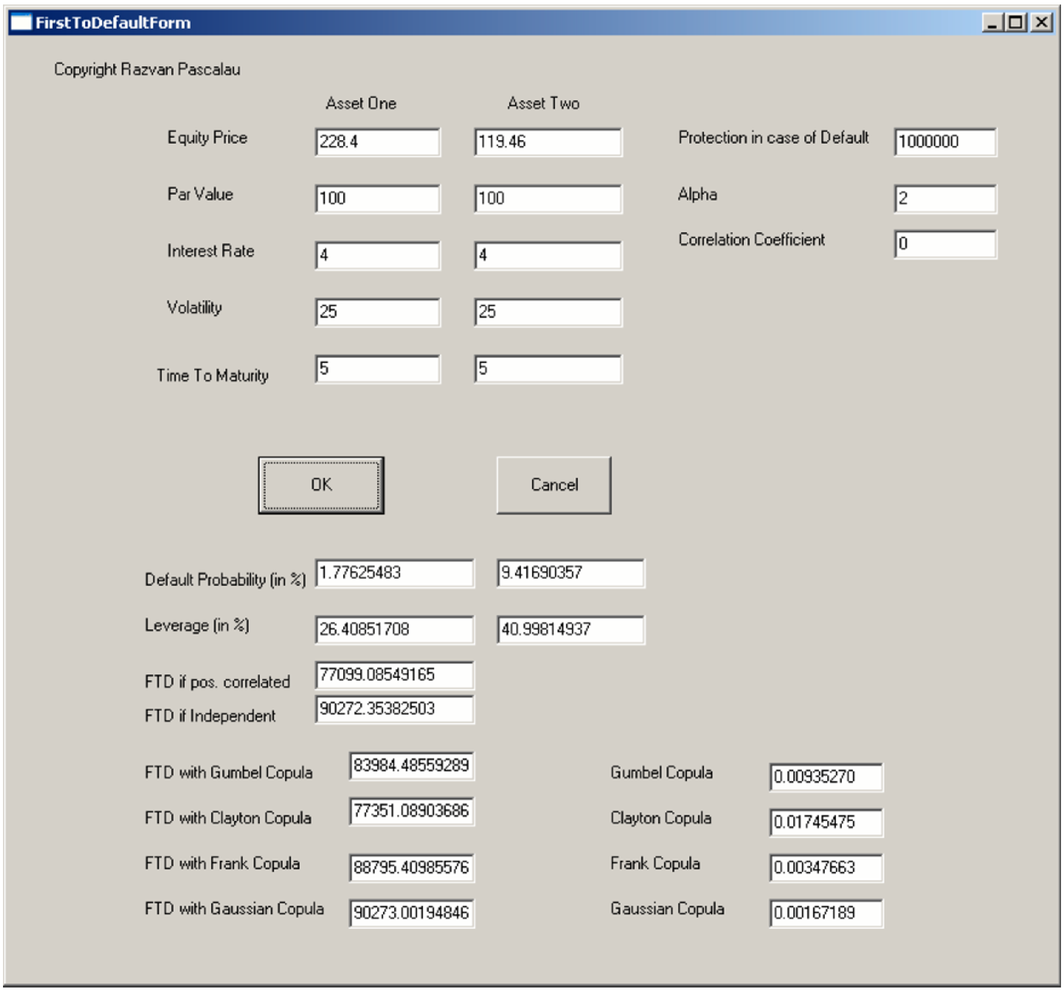

Figure 3.1: Caption Shot of the GUI

Below is a short excerpt of my source code:

```
FTDGumbel = inputProtection*exp(-(inputInterestRate2/100.0)*
           inputTimeToMaturity2)*(F.CalculateDefaultProb1()+
           F.CalculateDefaultProb2()-Gumbel);
FTDClayton = inputProtection*exp(-(inputInterestRate2/100.0)*
           inputTimeToMaturity2)*(F.CalculateDefaultProb1()+
           F.CalculateDefaultProb2()-Clayton);
FTDFrank = inputProtection*exp(-(inputInterestRate2/100.0)*
           inputTimeToMaturity2)*(F.CalculateDefaultProb1()+
           F.CalculateDefaultProb2()-Frank);
```

Bibliography

[1] Joe, H. *Multvariate Models and Dependence Concepts*. Chapman and Hall, 1997.

[2] Li, D.X. *On Default Correlation: A Copula Function Approach*. Journal of Fixed Income, 9, 43-54, 2000.

[3] Vechiatto et. al. *Copula Methods in Finance*. John Wiley & Sons, 2004.