Lecture 5

Volatility and ARCH Models

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EC 413

Read

RATS Handout Chapter 5

Why Volatility? What are the issues?

- Time varying risk premia
- Heteroskedastic variance
  - not constant variance
- News arrivals are serially (auto) correlated.
  - News tends to cluster in time
- Asymmetric reactions (leverage effects):
  - "People react more when prices fall."
- Non-linearity in the model
  - Time deformation (economic activity does not match calendar time)
- Leptokurtic distribution
  - Fat-tails and excess peakness at the mean

Volatility Models

1. **Moving Average Models**

m-day historic volatility estimate

$$\hat{\sigma}_t^2 = \frac{1}{m}\sum_{i=1}^{m} r_{t-i}^2$$

where r_{t-i} is the m most recent returns

$r_t = \Delta\log(P_t) = \log(P_t) - \log(P_{t-1})$: first difference of the price

data = growth rates of price

Questions:

- How to determine *m*?
- Equal weights for each term, r_{t-i}^2?
- Could we lose valuable information by smoothing out the series?

2. Exponentially Weighted Moving Averages (EWMA)

$$\hat{\sigma}_t^2 = \alpha\, r_{t-1}^2 + (1 - \alpha)\, \hat{\sigma}_{t-1}^2$$
$$= \alpha \sum_{i=1}^{\infty} (1 - \alpha)^{i-1}\, r_{t-i}^2$$

Note: Brooks (p. 443)

$$\hat{\sigma}_t^2 = (1-\lambda) \sum_{i=1}^{\infty} \lambda^{i-1} r_{t-i}^2$$

Note: This is the same as exponential smoothing method (except that y_t is replaced with r_{t-1}^2).

Remarks:

- If $\alpha = 1$, it's a naive model for volatility with r_{t-1}^2.
- If α is close to 1, recent values of r_{t-i}^2 are heavily weighted.
- The predicted volatility remains constant as the estimate at T.

3. Auto-Regressive Conditional Heteroskedasticity (ARCH) Models

"Express u_t^2 in terms of past values of u_t^2 (lagged squared residuals)."

$$u_t^2 = \omega + \alpha_1 u_{t-1}^2 + \ldots + \alpha_q u_{t-q}^2$$

Formally,

(1) Mean Equation: Usual ARMA models or others

$\mathbf{y_t = c + \phi\, y_{t-1} + u_t}$ .. AR(1) model, for instance

Let

$\mathbf{Var(u_t \mid \Omega_{t-1}) = h_t}$ .. conditional variance of u_t

where Ω_{t-1} is the information set available at t-1.

(2) Variance Equation: ARCH(Q) equation

$$\mathbf{h_t = \omega + \alpha_1 u_{t-1}^2 + \ldots + \alpha_q u_{t-q}^2}$$

This is an AR(1)-ARCH(Q) model!

More examples:

ARCH(2): $h_t = \omega + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2$
ARCH(3): $h_t = \omega + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \alpha_3 u_{t-3}^2$

4. GARCH (Generalized ARCH) Models

Additionally include <u>lagged variance terms</u>, h_{t-j}, j=1,..,P.

GARCH(P, Q): $h_t = \omega + \alpha_1 u_{t-1}^2 + \ldots + \alpha_q u_{t-q}^2$
$+ b_1 \mathbf{h_{t-1}} + \ldots + b_p \mathbf{h_{t-p}}$

Note: People often use GARCH (1,1) models, without having to search for optimal models.

GARCH(1, 1): $h_t = \omega + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1}$

<u>A complete specification of the AR(1)-GARCH(1,1) model</u>, for example:

$\mathbf{y_t = c + f\, y_{t-1} + u_t}$ .. AR(1) model, for instance

$\mathbf{Var(u_t \mid W_{t-1}) = h_t}$ .. conditional variance of u_t

$\mathbf{h_t} = w + a_1 \mathbf{u_{t-1}^2} + b_1 \mathbf{h_{t-1}}$ ... GARCH(1,1), for instance

More examples:

GARCH(1, 2): $h_t = \omega + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \beta_1 h_{t-1}$
GARCH(2, 1): $h_t = \omega + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1} + \beta_2 h_{t-2}$

Case Study: WPI Inflation

Questions:

1. <u>Why conditional?</u>
2. Time-varying volatility? Non-Constant variance?
3. Time-varying risk premia?
4. Is Volatility AUTO-CORRELATED?
5. Can we estimate/forecast the time-varying volatility?

<u>Conditional versus Unconditional Expectation</u>

(1) Mean

Example: AR(1) model

$$\mathbf{y_t = c +}\ f\ \mathbf{y_{t-1} + u_t}$$

Unconditional Mean:

$$\mu = E(y_t) = E(c + \phi\, y_{t-1} + u_t) = c + \phi\, \mu + 0 \quad \text{since } E(u_t) = 0$$

$$\text{Thus, } \mu = \frac{c}{1-\phi}$$

Conditional Mean:

$$E(y_t|\Omega_{t-1}) = c + \phi\, y_{t-1} + 0 \quad \text{since } E(y_t|\Omega_{t-1}) = 0$$

… c and y_{t-1} is included in $|\Omega_{t-1}$.

(2) Variance

Example: GARCH(1,1) model

$$\mathbf{h_t =} w + a_{\mathbf{1}}\mathbf{u_{t-1}^2} + b_{\mathbf{1}}\, \mathbf{h_{t-1}}$$

Unconditional Variance:

$$\sigma^2 = Var(u_t) = E(u_t^2) = \mathbf{E(}w + a_{\mathbf{1}}\mathbf{u_{t-1}^2} + b_{\mathbf{1}}\, \mathbf{h_{t-1}})$$

$$= \omega + a_{\mathbf{1}} E(\mathbf{u_{t-1}^2}) + b\ \mathbf{E(h_{t-1})} = \omega + a_{\mathbf{1}}\sigma^2 + b_{\mathbf{1}}\, \sigma^2$$

$$\text{Thus, } \sigma^2 = \frac{\omega}{1 - a_{\mathbf{1}} - b_{\mathbf{1}}}$$

Conditional Variance:

$$\mathbf{Var(u_t} \,|\, W_{\mathbf{t-1}} \mathbf{) = E(u_t^2} | W_{\mathbf{t-1}} \mathbf{) = h_t} = w + a_{\mathbf{1}}\mathbf{u_{t-1}^2} + b_{\mathbf{1}}\, \mathbf{h_{t-1}}$$

Estimating GARCH models

Main questions (as you may expect from ARMA models)

- How to choose P and Q?
- Testing for ARCH effects?
- It is a non-linear model, so how to estimate it?

Choosing P and Q

- Look at the ACF and PACF of squared residuals, u_t^2 of the ARMA model (see Case Study)
- Ljung-Box $Q^2(m)$ statistics using u_t^2 (see Case Study)
- AIC / SBC (can be done.)
- Your call: say, GARCH(1,1)

Note: If your GARCH model is optimal, the resulting squared residuals from GARCH models should look like a white noise process. (ACF/PACF, $Q^2(m)$ statistics, and other tests)

Testing for ARCH effects

LM test for ARCH effects.

From $h_t = \omega + \alpha_1 u_{t-1}^2 + \ \ldots + \alpha_q u_{t-q}^2$, we can have

H_0: $\alpha_1 = \alpha_2 = \ldots = \alpha_q = 0$.

Then, $h_t = \omega$ (constant!)

Thus, the testing hypothesis is

H_0: $\alpha_1 = \alpha_2 = \ldots = \alpha_q = 0$ (no ARCH effects)

1st. Run the usual **ARMA models**, and obtain the residual $\hat{u}_t$ ($\hat{u}_t^2$).

2nd. Regress $\hat{u}_t^2$ on $\hat{u}_{t-1}^2$, $\hat{u}_{t-2}^2$, …, $\hat{u}_{t-q}^2$

$$\hat{u}_t^2 = c + c_1\hat{u}_{t-1}^2 + c_2\,\hat{u}_{t-2}^2 + \quad + c_q\hat{u}_{t-q}^2 + \text{error}$$

LM $= T R^2 \sim \; c_q$ (chi-square with d.f. q.)

"Reject the null of no ARCH effects if LM > critical values."

Notes:

(i) It is important to test on the residuals from the mean equation (ARMA models), not from GARCH models.
(ii) We do not test directly for GARCH effects. If ARCH effects exist, GARCH models can be also considered

Example: Back to Case Study

Estimating GARCH models

We use the Maximum likelihood Estimation (MLE), since the GARCH model is non-linear. Using iterative algorithm such as BHHH or others, we search for optimal parameter values while maximizing the log likelihood function:

$$\text{Log Lik} = \sum_{t=1}^{T} \log f(y_i | W_{\mathbf{t-1}})$$ where f(.) is the density function.

Extensions of GARCH Models

1. GARCH-M model (GARCH in Mean model)

   Add h_t in the mean equation:

   $$y_t = c + \phi\, y_{t-1} + d\, \mathbf{h_t} + u_t$$

   If $\delta > 0$ and it is significant (t-test), there is a trade-off between the mean (return) and the conditional variance (time varying risk).

2. Leverage-GARCH (or also called, T-GARCH or Threshold-GARCH)

GJR model (Glosten, Jaganathan and Runkle, 1993)

We add an additional term which appears only when $u_{t-1} < 0$ (negative shock): Asymmetric effect

$$\mathbf{h_t} = w + a_1 \mathbf{u_{t-1}^2} + b_1 \mathbf{h_{t-1}} + \theta I_{t-1} \mathbf{u_{t-1}^2}$$

where $I_{t-1} = 1$ if $u_{t-1} < 0$ and $I_{t-1} = 0$ otherwise

If $\theta > 0$, we say that there is a leverage effect.

Bad news ($u_{t-1} < 0$) has an effect of $(\alpha_1+\theta)\, u_{t-1}^2$ on the variance.

Good news ($u_{t-1} \geq 0$) has an effect of $\alpha_1\, u_{t-1}^2$ on the variance.

If $\theta < 0$, vice versa.

One can test If $\theta = 0$ or $\theta > 0$ (t-test).

Example) Back to Case Study

Examine the coefficient of LEV.

3. E-GARCH Model (Exponential GARCH)

Two additions:

(i) Take a log form and
(ii) Add an additional term for a leverage effect (asymmetric effect).

$$\mathbf{\log h_t} = \; = \omega + \beta_1 \mathbf{\log h_{t-1}} + \alpha_1 [q \mathbf{V_{t-1}} + \gamma\{|V_{t-1}| - E|V_{t-1}|\}\ ]$$

where $\mathbf{V_{t-1}} = u_t/\sqrt{h_t}$

More questions:

(i) Why exponential? To guarantee Positive variance.
$h_t = \exp(\text{R.H.S.}) > 0$ always

(ii) Asymmetric effect

$q\mathbf{V_{t-1}} \rightarrow \theta < 0$ in usual cases

if $V_{t-1} < 0$, log h_t increases $\rightarrow h_t$ increases.

if $V_{t-1} > 0$, log h_t decreases $\rightarrow h_t$ decreases.

4. I-GARCH-M model (Integrated Garch)

Rearranging $h_t = \omega + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1}$, we can have

$u_t^2 = \omega + (\alpha_1 + \beta_1)\, u_{t-1}^2 + x_t - \beta_1 x_{t-1}$

with $x_t = u_t^2 - h_t$ such that $E(x_t) = E(u_t^2 - h_t) = E(x_{t-1}) = 0$

If $\alpha_1 + \beta_1 = 1$, we have

$u_t^2 = \omega + u_{t-1}^2$ (like a random walk model in variance!)

Then, the predicted variance is

$h_{t+s} = w + h_{t+s-1} = \ldots = w \bullet s + h_t$

$h_{t+s} = h_t$ if $w = 0$

This implies a persistence effect, as in the non-stationary (unit root) model.

This is an empirical phenomenon that we frequently observe in estimating GARCH models, especially in using high frequency data.

People who believe the I-GARCH model impose the restriction $\alpha_1 + \beta_1 = 1$, and estimate one of these parameters (α_1) and obtain the other ($\beta_1 = 1 - \alpha_1$).

5. Other Extensions

Component GARCH models
Multivariate GARCH models
Smooth Transition GARCH models (Lee..)
Many other Brothers and Sisters GARCH models..

Forecasting with GARCH Models

Example with GARCH(1,1)

$h_t = \omega + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1}$

with unconditional Variance $\sigma^2 = \dfrac{\omega}{1 - \alpha_1 - \beta_1}$

Write:

$h_t = \sigma^2 + \alpha_1 (u_{t-1}^2 - \sigma^2) + \beta_1 (h_{t-1} - \sigma^2)$

$h_{t+s} = \sigma^2 + \alpha_1 (u_{t+s-1}^2 - \sigma^2) + \beta_1 (h_{t+s-1} - \sigma^2)$

Then, predicted h_{t+s} is:

$h_{t+s} = \omega + (\alpha_1 + \beta_1) (h_{t+s-1} - \sigma^2)$

Review Questions on ARCH-GARCH Models

1. What are major limitations of using MA or EWMA models for volatility?

2. What are economic reasoning for (i) conditional and (ii) heteroskedasticity (not constant variance) implied in the ARCH-GARCH models.

3. Briefly explain about the GARCH-M model and underlying reasons for this model.

4. Briefly explain about the T-GARCH model and underlying reasons for this model.

5. Briefly explain about the E-GARCH model and underlying reasons for this model.

6. Discuss how you can test for (existence of) ARCH effects.

7. Using the excess rates of returns, which was defined as the difference between the rate of returns of interest rates and the rate of return of risk free assets (Treasury Bill), estimate a GARCH model of your choice. Note that you do not need to difference the data. (File: excess.txt)

(a) Identify the orders, p and q, of an ARMA(p,q) model. Use AIC for this.
(b) Using the residuals of the identified ARMA model estimation result, test for whiteness of residuals. Use Q-statistics at 5 and 10 lags. Also, check ACF and PACF of residuals.
(c) Using the residuals of the identified ARMA model estimation result, test for ARCH effect. Use 5 lags for this test.
(d) Estimate a GARCH(1, 1) model.
(e) Using the squared residuals of the estimated GARCH model, test for any further ARCH effect. Use Q-statistics at 5 and 10 lags.
(f) Plot the estimated conditional heteroskedasticity.
(g) Find the unconditional mean and the unconditional variance from the ARMA-GARCH model that you have estimated.