Lecture 4

Eigenvalue Problem and Quadratic Forms

Read Ch 10

<u>Def</u> linear independence of two vectors

Two vectors, $\underset{(n\times 1)}{u}$ and $\underset{(n\times 1)}{v}$, are <u>linearly independent</u>

if $\lambda_1 u + \lambda_2 v = \underset{(n\times 1)}{\mathbf{0}}$ only when $\lambda_1 = \lambda_2 = 0$.

Eg) Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. u & v are <u>independent</u>. why?

$$\lambda_1 u + \lambda_2 v = \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is zero only when $\lambda_1 = \lambda_2 = 0$.

$$\underset{(\lambda_1 = 0)}{0} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \underset{(\lambda_2 = 0)}{0} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

There are no other values of λ_1 & λ_2 to make $\lambda_1 u + \lambda_2 v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Eg2) Let $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$$\lambda_1 u + \lambda_2 v = \lambda_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ 2 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

yes, if $\lambda_1 = -2$ and $\lambda_2 = 1$.

$$-2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, u and v are linearly <u>dependent</u>

Eg3) Let $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\lambda_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

this is zero only if $\lambda_1 = \lambda_2 = 0$

thus, u & v are <u>linearly</u> <u>independent</u>.

<u>Exercise</u> Are they linearly independent?

(a) $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(b) $u = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} -5 \\ -5 \end{pmatrix}$

<u>Def</u> Two <u>vectors</u>, u and v, are <u>orthogonal</u>

if $u'v = 0$.

eg) $u = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$ $v = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$

$$u'v = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)\begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix} = \left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}}\right)\left(\frac{-2}{\sqrt{5}}\right) = \underset{(1\times 1)}{0}$$

thus, u & v are orthogonal.

<u>Def</u> Rank of a <u>matrix</u> is the number of linearly independent columns (or rows).

eg) $A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 4 \end{bmatrix}$ ⇐ Rows are all linearly independent (rank = 3)

4th column = 2 x 1st column
they are linearly dependent.

(rank = 3)

Max of column rank and row rank
(min)
= rank of A

<u>Note</u> A (n×n) matrix A is nonsingular (ie A^{-1} exists).
if $\det(A) = |A| \neq 0$ and $\text{rank}(A) = n$
(full rank)

<u>point</u>: If columns or rows are <u>linearly dependent</u>
so that $\text{rank}(A) < n$, A^{-1} does <u>not</u> exist.

Ex) Find the rank of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Eigenvalue problem

<u>Def</u> Consider an equation that λ and x satisfy: *
(other than $x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$)

$$\boxed{Ax = \lambda x}$$

where A = n×n matrix
x = n×1 vector
λ = scalar

Then x = eigen vector (characteristic vector)
λ = eigen value. (characteristic root)

eg1)
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the eigenvector

$\lambda = 1$ is the eigenvalue.

Eg2) Find the eigenvalue and eigenvector of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} = 0 \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

same vector.
what is this?

Eg3) Find the eigenvalue & eigenvector of

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

$x = \begin{pmatrix} \\ \end{pmatrix}$ $\lambda = ?$

(Formal way to find them)

$Ax = \lambda x$ $(A = n \times n,\ x = n \times 1,\ \lambda = 1 \times 1)$

$\Rightarrow Ax - \lambda x = 0 \Rightarrow (A - \lambda I)x = 0$ (why include I?)

$\Rightarrow |A - \lambda I| = 0$ since $x \neq 0$

: characteristic equation

that is, find λ to satisfy $|A - \lambda I| = 0$

eg) $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

$$(A-\lambda I) = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) - 4 \overset{\text{let}}{=} 0$$

$$\Rightarrow \lambda^2 - 5\lambda = 0$$

$$\Rightarrow \lambda(\lambda-5) = 0$$

$$\Rightarrow \underline{\lambda = 0 \text{ or } 5} \quad \text{(two eigenvalues)}$$

How?

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad ; \quad \lambda = 0$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad ; \quad \lambda = 5$$

Here $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 0$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 5$.

Two questions

i) How did you find these eigenvectors?

ii) Are they unique?

How about?

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 20 \\ 10 \end{pmatrix} = 5\begin{pmatrix} 20 \\ 10 \end{pmatrix}$$

these are valid eigenvectors, too.

Also $\begin{pmatrix} 10 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \ldots$ and so on.

<u>Note</u> Since eigenvectors are not unique, we wish to <u>normalize</u> eigenvectors such that $x_1^2 + x_2^2 = 1$.

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ = eigenvector

since $\boxed{(A - \lambda I)\, x = 0}$

$$\left[\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $\begin{pmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i) when $\lambda = 5$

$$\begin{pmatrix} 4-5 & 2 \\ 2 & 1-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} -x_1 + 2x_2 = 0 \\ 2x_1 - 4x_2 = 0 \end{matrix}\right) \Rightarrow x_1 = 2x_2$$

use $x_1^2 + x_2^2 = 1$. $(2x_2)^2 + x_2^2 = 1$

$$x_2^2 = \frac{1}{5} \Rightarrow x_2 = \pm \frac{1}{\sqrt{5}}$$

choose $x_2 = \frac{1}{\sqrt{5}}$. Then $x_1 = \frac{2}{\sqrt{5}}$ since $x_1 = 2x_2$

thus $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$ is the eigenvector.
$\underset{\text{let}}{=} g_1$

ii) when $\lambda = 0$

$$\begin{pmatrix} 4-0 & 2 \\ 2 & 1-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 4x_1 + 2x_2 = 0 \\ 2x_1 + x_2 = 0 \end{matrix}$$

or $x_2 = -2x_1$

using $x_1^2 + x_2^2 = 1$

$\Rightarrow x_1^2 + (-2x_1)^2 = 1 . \quad 5x_1^2 = 1 .$

$x_1^2 = \frac{1}{5} , \quad x_1 = \pm \frac{1}{\sqrt{5}}$

choose $\frac{1}{\sqrt{5}}$. Then $x_2 = -2x_1 = -2\left(\frac{1}{\sqrt{5}}\right) = \frac{-2}{\sqrt{5}}$

thus $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{pmatrix} \overset{\text{let}}{=} q_2$

therefore, eigenvectors are

$$Q = [q_1, q_2] = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$$

$\lambda = 5 \quad \lambda = 0$

Note q_1 and q_2 are orthogonal.

$q_1' q_2 = 0$ (eigenvectors are orthogonal)

Note $q_1' q_1 = 1, \quad q_2' q_2 = 1.$ why?

Note $Q = (q_1, q_2)$ is an orthogonal matrix

$$Q'Q = QQ' = I$$

check $\underset{Q}{\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix}} \underset{Q'}{\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix}} = \underset{I}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

Note when one of eigenvalues is zero, the matrix is singular.

ex) Find the eigenvalues & eigenvectors of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

i) From $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 2\lambda = 0$$

or $\lambda(\lambda - 2) = 0$

thus $\lambda = 0$ or 2 : eigenvalues

ii) From $(A - \lambda I)x = 0$

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

when $\lambda = 0$

$$\begin{pmatrix} 1-0 & 1 \\ 1 & 1-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 + x_2 = 0$$

or $x_1 = -x_2$

using $x_1^2 + x_2^2 = 1$, $(-x_2)^2 + x_2^2 = 1$ or $2x_2^2 = 1$

$\Rightarrow x_2 = \pm \frac{1}{\sqrt{2}}$. choose $x_2 = \frac{1}{\sqrt{2}}$ $\Rightarrow x_1 = -x_2 = \frac{-1}{\sqrt{2}}$

thus, $q_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

when $\lambda = 2$

$$\begin{pmatrix} 1-2 & 1 \\ 1 & 1-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -x_1 + x_2 = 0$$

or $x_1 = x_2$

using $x_1^2 + x_2^2 = 1$, $x_1^2 + x_1^2 = 1$ or $2x_1^2 = 1$

$\Rightarrow x_1 = \pm \frac{1}{\sqrt{2}}$. choose $x_1 = \frac{1}{\sqrt{2}}$ $\Rightarrow x_2 = x_1 = \frac{1}{\sqrt{2}}$

thus $q_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

then $Q = (q_1, q_2) = \left[\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right]$

Also we can check $q_1' q_2 = 0$, $q_1' q_1 = q_2' q_2 = 1$

and $Q'Q = QQ' = I$. (Do this!)

<u>Exercise</u> p. 435 Q. 1

Find the eigenvalues & eigenvectors of

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

<u>Exercise</u> H/W

a) Find the eigenvalues & eigenvectors of

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

b) Q. 3, p. 435 (Find the eigenvalues of $X'X$ (not P.))

<u>Spectral decomposition</u> (p. 424)

We know that the orthogonal matrix Q satisfies

$Q'Q = QQ' = I$ where Q = eigenvector matrix

This implies that $Q' = Q^{-1}$.

Then consider a matrix of eigenvalues

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

(If A is non-symmetric $Q^{-1} A Q = \Lambda$.)

Then $\boxed{Q' A Q = \Lambda}$ if A is symmetric.

Since $Q' = Q^{-1}$, premultiply Q and postmultiply Q' to both sides

$\boxed{A = Q \Lambda Q'}$ if A is symmetric

<u>Note</u> If A is not symmetric, these results may not hold.

Also

there exists a matrix P such that

$$\boxed{A = PP' \quad \text{with} \quad P = Q\Lambda^{\frac{1}{2}}}$$

if A is symmetric and positive semidefinite.

Eg) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $(\lambda=1)$ $(\lambda=3)$

We found that $\lambda = 1, 3$ and $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

checking $Q'AQ = \Lambda$

$$Q'AQ = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \overset{\text{it is}}{=} \Lambda$$

checking $Q\Lambda Q' = A$

$$Q\Lambda Q' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \cdots = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \overset{\text{it's}}{=} A$$

Finding P s.t. $A = PP'$

$$P = Q\Lambda^{\frac{1}{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \end{pmatrix}$$

checking $PP' = A$

$$PP' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{\sqrt{3}}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \overset{\text{it's}}{=} A$$

Exercise) Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

a) Find eigen values and eigenvectors.

b) Is it a symmetric matrix? Is it p.d.?

c) Find P such that $PP' = A$

d) check $Q'AQ = \Lambda$, $Q\Lambda Q' = A$ & $PP' = A$

Quadratic Forms

<u>Def</u> Given an $n \times n$ matrix A, and $n \times 1$ vector x, the quadratic form (scalar) function is defined as

$$q(x) = x'Ax$$

<u>Note</u> $x'Ax = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$

eg) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$; 2×2 $\quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$: 2×1

$$q(x) = x'Ax = (x_1, x_2) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (2x_1 + x_2, x_1 + 2x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= 2x_1^2 + x_1x_2 + x_1x_2 + 2x_2^2 \quad (\text{coef} = 2, 1, 1, 2)$$

this is a scalar : 1×1.

<u>Note</u> If A is not symmetric, we can have A^* with its elements being

$$a_{ij}^* = a_{ji}^* = \frac{1}{2}(a_{ij} + a_{ji})$$

so that the quadratic form of $q(x)$ using A^* is the same as that of A.

Definiteness

Def If $q(x) = x'Ax > 0$ for all $x \neq 0$, $q(x)$ is positive definite. (if "$\geq$", positive semi-definite), and A is a positive definite (p.d) matrix.

Note If "$<$", $q(x)$ is negative definite, and A is a negative definite matrix.
(if "$\leq$", negative semi-definite)

There are two ways to check definiteness

(1) thm : If all eigenvalues are positive, a symmetric matrix A is positive definite.
(If positive and zero, positive semi-definite)

proof) For a matrix A, we can have $Q'AQ = \Lambda$.
using Q, we use $x = Qy$.

$$x'Ax = y'\underbrace{Q \cdot A \cdot Q}y = y'\Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2$$

which is positive if all λ_i (eigenvalues) are positive.

(2) thm : If the determinant of every leading principal submatrix is positive, a symmetric matrix A is positive definite.

(if positive and zero, positive semi-definite)

Note leading principal submatrix, H_K

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$H_1 = [a_{11}], \quad H_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad H_3 \overset{A}{=} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Note leading principal minors = $|H_K|$ = determinants.

eg) $|H_1| = a_{11}$, $|H_2| = a_{11}a_{22} - a_{21}a_{12}$

$|H_3| = |A|$

Ex1) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

(i) eigenvalues $= 1, 3$.

thus A is p.d.

(ii) $|H_1| = 2 > 0$, $|H_2| = |A| = 2\times 2 - 1\times 1 = 3 > 0$

thus A is p.d.

Ex2) $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

$|H_1| = 4 > 0$, $|H_2| = 8 - 0 > 0$.

thus A is p.d.

Ex3) $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ $\quad$ A is p.d.

why?

ex4) $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

i) eigenvalues $= 0, -2$

thus, A is negative semi-definite

ii) $|H_1| = |-1| < 0$, $|H_2| = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0$

This result shows that A is negative semi-definite.

Thm If principal minors "alternate" in sign starting with negative, a symmetric matrix is negative definite.

(if some of principal minors are zero, and others alternate in sign starting with negative, it's negative semi-definite.)

e5) $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$|H_1| = |-1| < 0$, $|H_2| = 0$.

the principal minors alternate in sign starting with negative $\Rightarrow$ A is negative semi-definite (since $|H_2| = 0$)

Note If $|H_k| = 0$, all minors $|H_k^*|$ need to be checked. (See p.506, ch 11)

Ex1) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$|H_1| = -2 < 0$

$|H_2| = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4 - 1 > 0$

the sign alternate starting with negative.

thus, A is negative definite.

Ex2) $A = \begin{bmatrix} -7 & -1 \\ -1 & 2 \end{bmatrix}$

$|H_1| = -7 < 0$

$|H_2| = \begin{vmatrix} -7 & -1 \\ -1 & 2 \end{vmatrix} = -14 - 1 = -15 < 0$

the sign does <u>not</u> alternate.

Thus, A is neither p.d. nor n.d.

Ex3) $A = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}$

$|H_1| = -2$

$|H_2| = |A| = (-2)(-5) - (2)(5) = 0$

since $|H_2| = 0$, we need to check all minors

$|H_1^*| = \underset{-2}{\underset{\parallel}{|a_{11}|}}$ or $\underset{-5}{\underset{\parallel}{|a_{22}|}}$. Both are negative.

we need to check both $|H_1^*|$.

$|H_2^*| = |A| = 0$.

thus A is negative semi-definite

Ex 4) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix}$

$|H_1| = 1 > 0, \quad |H_2| = \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 4-1 = 3 > 0$

$$|H_3| = 1\cdot C_{11} + 1\cdot C_{12} + 0\cdot C_{13}$$

$$= 1\cdot\begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix}$$

$$= 8 - 3 + 0 = 5 > 0$$

Thus, A is p.d., when all $|H_k| > 0$.

Exercise Determine the definiteness. (H/W)

(a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

Exercise (H/W)

(a) $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ Determine the definiteness.

(b) Review Exercises p 451 Q 7 (eigenvalues), Q.9 (orthogonal decomposition)

(c) Review questions p 450 Q.5 ~ Q.12

Application of Eigen-value problems

(1) Multivariate Analysis of Variance (MANOVA)

$$\text{Max} \quad \frac{x'Hx/(k-1)}{x'Ex/(n-k)} \qquad x: p\times 1$$

$$\text{or Max} \quad x'Hx \quad \text{s.t.} \quad x'Ex = C$$

$$\mathcal{L} = x'Hx - \lambda(x'Ex - C)$$

$$d\mathcal{L}/dx = 2Hx - 2\lambda Ex = 0$$

$$(H - \lambda E)\,x = 0$$

$$\text{or } \left[E^{-1}H - \lambda I\right] x = 0$$

$$\text{or } (A - \lambda I)\,x = 0 \quad \text{with } A = E^{-1}H$$

- the largest eigenvalue is the solution to this max. problem.
- The max possible F-ratio is obtained the linear combination of the original p-variables. The eigenvector gives the linear combination.

Note E (within) $\underset{p\times p}{E} = \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})(y_{ij} - \bar{y}_{i.})'$

H (between) $\underset{p\times p}{H} = n\sum_{i=1}^{k}(\bar{y}_{i.} - \bar{y}_{..})(\bar{y}_{i.} - \bar{y}_{..})'$

(2) Canonical Correlation Analysis

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix}$$ 2 sets of variables.

want to find a correlation coefficient between 2 <u>sets</u> of variables.

<u>Note</u> $\mathrm{Corr}(x, y) = \dfrac{S_{xy}}{S_x S_y}$ between 2 variables.

Consider

$$S_{XX}^{-1} S_{XY} S_{YY}^{-1} S_{YX} \quad \text{or} \quad S_{YY}^{-1} S_{YX} S_{XX}^{-1} S_{XY}$$

and find eigenvalues of these matrices

$$\left| \underbrace{S_{XX}^{-1} S_{XY} S_{YY}^{-1} S_{YX}}_{A} - \lambda I \right| = 0$$

i) $\lambda = r^2$ where r = canonical correlation coeff
= largest eigenvalue

ii) Eigenvectors give the linear combination of 2 sets of variables to maximize the correlation coeff. (canonical correlation coeff.)

(3) Principal Component.

Let S be the sample variance of p-variables, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}$

$$S = \begin{bmatrix} s_1^2 & s_{12} & \cdots & s_{1p} \\ & s_2^2 & \cdots & s_{2p} \\ & & \ddots & \\ & & & s_p^2 \end{bmatrix}$$

consider

$$[S - \lambda I] x = 0$$

λ = eigenvalue of S

x = eigenvector of S

principal component

$$z_1 = g_1' y, \quad z_2 = g_2' y, \quad \ldots \quad z_p = g_p' y$$

1st p.c   2nd p.c   p-th p.c.

why?

$$\max_x \; x' S x \quad \text{s.t} \quad x'x = 1$$

$$\mathcal{L} = x' S x - 2\lambda (x'x - 1)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2Sx - 2\lambda x = 0$$

$$(S - \lambda I) x = 0$$