(Part IV)

Lecture 9

Dynamic Models

Read Hoy
- ch 17
- ch 18 (18.1)] also (ch. 19.1)
- ch 20 (20.1)]
- ch 21 (21.1)] skip ch22, ch24
- ch 23 (23.1)]
- ch 25 (25.1, 25.2, 25.3, 25.4)

Topics
- Difference Equations
- Differential Equations
- Optimal Control Theory (ch25)

Ch17

Distinguish

Difference equation (discrete time context)

eg) $p_{t+1} - p_t = \theta p_t$ .. 1st order, autonomous (no t)

$y_t = 2y_{t-1} + 3y_{t-2} + 2t$ .. 2nd order (2 lags), nonautonomous (2t, ..)

Differential equation (continuous time context)

eg) $\dot{y} + 2y = 8$ where $\dot{y} = \frac{dy}{dt}$ ... 1st order autonomous

$3\ddot{y} + 3\dot{y} + y = 5t$ where $\ddot{y} = \frac{d^2y}{dt^2}$ .. 2nd order nonautonomous

$\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2$ ... 3rd order autonomous

$\dot{k} = I(t) - \delta k(t)$ where $\dot{k} = \frac{dk}{dt}$ .. 1st order

Terms

order, autonomous, solution
(above)

eg) $y_{t+1} = 2y_t$, $t = 0, 1, 2 \ldots$ with $y_0 = 1$

.. 1st order difference equation.

solution : $y_t = 2^t$

why? $y_1 = 2y_0 = 2$, $y_2 = 2\times 2 = 4$, $y_3 = 2\cdot 4 = 8$

$y_4 = 2y_3 = 2\times 8 = 16 = 2^4$, ...

eg) $\dot{y} = b$ with $y(0) = 3$

solution : $y(t) = bt + 3$

why $\frac{dy(t)}{dt} = b$ $\quad y(t) = bt + C$

$y(0) = 0 + C \Rightarrow C = 3$

1st order difference equation (linear)

$$y_{t+1} = a y_t + b \quad , \; t = 0, 1, 2 \ldots$$

want to find a solution:

$t=0$: $y_1 = a y_0 + b$

$t=1$; $y_2 = a y_1 + b = a(a y_0 + b) + b = a^2 y_0 + ab + b$

$t=2$: $y_3 = a y_2 + b = a(a^2 y_0 + ab + b) + b = a^3 y_0 + a^2 b + ab + b$

$\vdots$

$y_t = a^t y_0 + \underbrace{a^{t-1} b + \cdots + ab + b}$

$\hookrightarrow = b(1 + a + \cdots + a^{t-1}) = b\left(\frac{1-a^t}{1-a}\right)$

thus

$$y_t = \begin{cases} a^t y_0 + b\left(\frac{1-a^t}{1-a}\right) & \text{if } a \neq 1 \\ y_0 + bt & \text{if } a = 1 \quad \text{(unit root)} \end{cases}$$

Eg1) Deposit \$100. 10% annual interest. after t-th year,

$y_t = 100(1.1)^t \Leftrightarrow y_{t+1} = 1.1 y_t$

(solution)

Eg2) Deposit \$100 every year. 10% annual interest. after t-th year

$y_{t+1} = 1.1 y_t + 100$ ← drift $\quad (y_0 = 100)$

$\Rightarrow$ solution $\quad y_t = 100(1.1)^t + 100\left(\frac{1-1.1^t}{1-1.1}\right)$

If y_0 is unknown, the solution is

$y_t = C(1.1)^t + 100\left(\frac{1-1.1^t}{1-1.1}\right)$ ... general solution

In general, for the 1st order difference eq.

$$\boxed{y_t = \begin{cases} \underbrace{c\,a^t}_{y_c} + \underbrace{b\left(\frac{1-a^t}{1-a}\right)}_{y_p} & \text{if } a \neq 1 \\ c + bt & \text{if } a = 1 \end{cases}}$$

y_c complementary solution (y_0)

y_p particular solution

eg) $y_{t+1} = 0.5 y_t + 10$ with $y_0 = 1$

$$y_t = c(0.5)^t + 10\left(\frac{1-0.5^t}{1-0.5}\right)$$

From $y_0 = 1$, $t = 0$

$$y_0 = c(0.5)^0 + 10 \cdot \frac{1-0.5^0}{1-0.5} = c \Leftarrow 1 = y_0$$

Thus $c = 1$.

$$y_t = 1 \cdot (0.5)^t + 10\left(\frac{1-0.5^t}{1-0.5}\right)$$

$\Rightarrow$ y_c can be specified if y_0 is given.

Exercise $y_{t+1} = 5 y_t + 1$, $y_0 = \frac{7}{4}$

ans) $y_t = 2(5)^t - \frac{1}{4}$ Find this!

Exercise a) $2y_{t+1} = y_t + 6$, $y_0 = 7$

b) $y_{t+1} = 0.2 y_t + 4$, $y_0 = 4$

Ex) Cobweb model of price adjustment

$$q^D = A + BP_t$$

$$q^S = F + G \cdot E_{t-1}(P_t) = F + G \cdot P_{t-1}$$

where $E_{t-1}P_t$ is the expected price

; $E_{t-1}P_t = P_{t-1}$ (best prediction is yesterday price)

$q^D = q^S$ gives

$$\Rightarrow A + BP_t = F + GP_{t-1}$$

$$\Rightarrow P_t = \left(\frac{G}{B}\right) P_{t-1} + \left(\frac{F-A}{B}\right)$$

$$\Rightarrow P_t = a P_{t-1} + b \quad \text{where } a = \frac{G}{B},\ b = \frac{F-A}{B}$$

Solution

$$P_t = c\,a^t + b\left(\frac{1-a^t}{1-a}\right) \rightarrow \frac{b}{1-a} \quad \text{assuming } a = \frac{G}{B} < 1$$

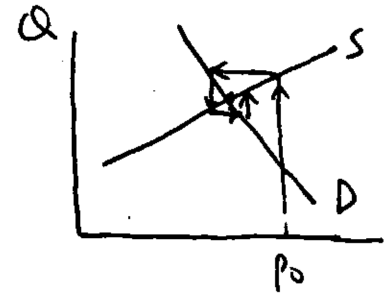
Note $|a|<1$ implies $|G|<|B|$.

B = slope of demand function < 0

G = slope of supply function

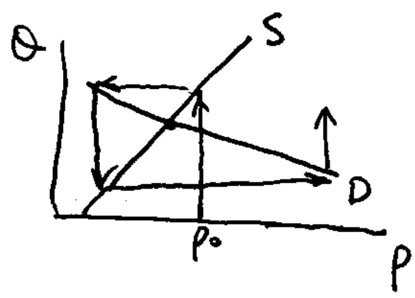
thus, supply function is flatter.

(phase diagram)



It converges!

Note $|a|>1$. Demand function is flatter



It diverges! (explosive)

Converges to what?

(ignore t)

· $p_t = a p_{t-1} + b \Rightarrow \bar{p} = a\bar{p} + b \Rightarrow \bar{p} = \frac{b}{1-a}$

$\Rightarrow \bar{p} = \dfrac{\frac{F-A}{B}}{1-\frac{G}{B}} = \dfrac{A-F}{G-B}$

same

· $p_t = c a^t + b\left(\frac{1-a^t}{1-a}\right) \rightarrow \frac{b}{1-a}$ since $a^t \rightarrow 0$ if $|a|<1$

(let $a^t \rightarrow 0$)

exercises a) Ex 18. Review question p787, Q.5, Q.6

b) Ex 18. " p786. Q.1 (Do this first)

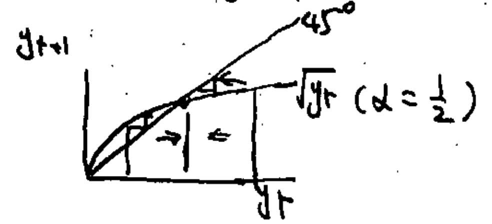
$\boxed{\text{ch19}}$

First order difference equation (nonlinear)

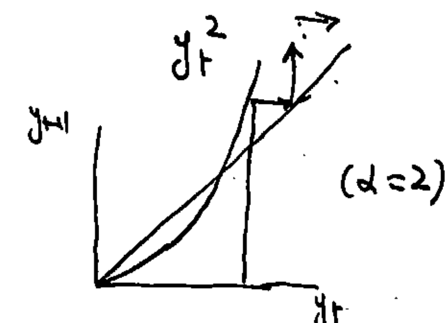
$y_{t+1} = f(y_t)$

nonlinear function

eg) $y_{t+1} = y_t^{\alpha}$ $\quad \alpha \neq 1$



(converges)



(diverges)

This is a qualitative graphic approach

(45° = intertemporal equilibrium $y_{t+1} = y_t$)

(skip)

Second-order difference equations

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = b, \quad t=0,1,2,\dots$$

Solution: $y_t = y_c + y_p$

y_c: complementary (general) solution $\downarrow$ (from homogeneous function)

y_p: particular solution $\downarrow$ (long-run level $y_c = \bar{y}$ for any t)

Note If $b=0$, it's a homogeneous equation

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

(1) General (complementary) solution

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0, \quad t=0,1,2,\dots$$

i) if $a_1^2 - 4a_2 > 0$ (2 roots)

$$\boxed{y_c = c_1 r_1^t + c_2 r_2^t}$$

ii) if $a_1^2 - 4a_2 = 0$ (1 root)

$$\boxed{y_c = c_1 r^t + c_2 t r^t = (c_1 + c_2 t) r^t}$$

iii) if $a_1^2 - 4a_2 < 0$ (complex root)

$$\boxed{y_c = a_2^{t/2}(c_1 \cos \theta t + c_2 \sin \theta t)}$$

where $r_1, r_2 = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$

Ex1) $y_{t+2} - 6y_{t+1} + 8y_t = 0$

characteristic equation

$$r^2 - 6r + 8 = 0 \Rightarrow r = 2, 4$$

Thus

$$y_t = c_1 2^t + c_2 4^t$$ .. this is the complementary (general) solution, y_c.

Here, c_1 & c_2 can be calculated if initial values (y_0 and y_1) are given.

If $y_0 = 1$, $y_1 = 3$, then

$$y_t = c_1 2^t + c_2 4^t \Rightarrow y_0 = c_1 \cdot 2^0 + c_2 \cdot 4^0 = 1 \Rightarrow c_1 + c_2 = 1$$
$$y_1 = c_1 \cdot 2^1 + c_2 \cdot 4^1 = 3 \Rightarrow 2c_1 + 4c_2 = 3$$

$\Rightarrow$ Solving them simultaneously

$$c_1 = 2 \times 1 - \frac{1}{2} \times 3 = \frac{1}{2}, \quad c_2 = -1 + \frac{1}{2} \times 3 = \frac{1}{2}$$

Thus

$$y_c \Rightarrow y_t = \left(\frac{1}{2}\right) 2^t + \left(\frac{1}{2}\right) 4^t.$$

Ex2) $y_{t+2} + y_{t+1} - 2y_t = 0$ with $y_0 = 4$, $y_1 = 5$

$$r^2 + r - 2 = 0 \Rightarrow r = 1, -2$$

$$y_c: \quad y_t = c_1 (1)^t + c_2 (-2)^t$$

$$y_0 = c_1 (1)^0 + c_2 (-2)^0 = 4 \Rightarrow c_1 + c_2 = 4$$
$$y_1 = c_1 (1)^1 + c_2 (-2)^1 = 5 \Rightarrow c_1 - 2c_2 = 5$$

Thus $c_1 = \dfrac{\begin{vmatrix} 4 & 1 \\ 5 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = \dfrac{13}{3}$, $c_2 = \dfrac{\begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix}}{-3} = -\dfrac{1}{3}$

$$y_t = \frac{13}{3}(1)^t + \left(-\frac{1}{3}\right)(-2)^t$$

ex3) $y_{t+2} - 4y_{t+1} + 4y_t = 0$

$$r^2 - 4r + 4 = 0 \Rightarrow r = 2 \quad \text{(double roots)}$$

$$y_c: \quad y_t = (c_1 + c_2 t)2^t$$

Note why characteristic equation?

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0 \Rightarrow r^2 + a_1 r + a_2 = 0$$

$$r_1, r_2 = \frac{-a \pm \sqrt{a_1^2 - 4a_2}}{2}$$

The roots can be found as eigenvalues of

$$A = \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix} \Rightarrow |A - rI| = 0$$

$$\Rightarrow \begin{vmatrix} a_1 - r & a_2 \\ 1 & 0-r \end{vmatrix} = 0$$

$$\Rightarrow (a_1 - r)(-r) - a_2 = 0 \Rightarrow r^2 + a_1 r + a_2 = 0$$

In general, for the p-th order difference equation the roots can be found from $|A - rI| = 0$, where

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots a_{p-1} & a_p \\ 1 & 0 & 0 & \cdots 0 & 0 \\ 0 & 1 & 0 & \cdots 0 & 0 \\ 0 & 0 & 1 & \cdots 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots 1 & 0 \end{bmatrix} : AR(1) \text{ companion form}$$

eg)

$$y_{t+3} + a_1 y_{t+2} + a_2 y_{t+1} + a_3 y_t = 0$$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow |A - rI| = 0$$

r can be found!

eg1)

$$y_{t+2} = -a_1 y_{t+1} - a_2 y_t$$

$[Y_t = A \cdot Y_{t-1}]$

$$\begin{pmatrix} y_{t+2} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} -a_1 & -a_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{t+1} \\ y_t \end{pmatrix}$$

$$|A - rI| = 0 \Rightarrow r^2 + a_1 r + a_2 = 0$$

eg2)

$$y_{t+3} = -a_1 y_{t+2} - a_2 y_{t+1} - a_3 y_t$$

$$\begin{pmatrix} y_{t+3} \\ y_{t+2} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_{t+2} \\ y_{t+1} \\ y_t \end{pmatrix}$$

$[Y_t = A \cdot Y_{t-1}]$

(2) Particular Solution

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = b$$

Let $y_{t+s} = \bar{y}$ (ignore t)

$$\bar{y} + a_1\bar{y} + a_2\bar{y} = b$$

$$\Rightarrow \bar{y} = \frac{b}{1+a_1+a_2}$$

$\Rightarrow$ Complete Solution

$$= y_c + y_p$$

ex) $y_{t+2} + 4y_{t+1} + 3y_t = 16$, $y_0 = 1$, $y_1 = 2$

i) complementary solution

$$r^2 + 4r + 3 = 0 \Rightarrow r = -1, -3.$$

$$y_c : y_t = c_1(-1)^t + c_2(-3)^t$$

ii) particular solution

$$\bar{y} + 4\bar{y} + 3\bar{y} = 16 \Rightarrow \bar{y} = 2 = y_p$$

Thus, the complete solution is

$$y_t = y_c + y_p = c_1(-1)^t + c_2(-3)^t + 2$$

Now, $y_0 = 1$, $y_1 = 2$

$$y_0 = c_1(-1)^0 + c_2(-3)^0 + 2 = 1 \Rightarrow c_1 + c_2 = -1$$

$$y_1 = c_1(-1)^1 + c_2(-3)^1 + 2 = 2 \Rightarrow -c_1 - 3c_2 = 0$$

thus $c_1 = -\frac{3}{2}$, $c_2 = \frac{1}{2}$

$$\Rightarrow y_t = \underbrace{\left(-\frac{3}{2}\right)(-1)^t + \frac{1}{2}(-3)^t}_{y_c} + \underbrace{2}_{y_p}$$

Exercises H/W

i) Ex 20.2, p.843 Q.1 (a), Q.2 (a). (skip this.)

ii) Review Ex. 20, p845 Q1, 2, 3, Q.12

* Exercise Ex 20.1, p.836 Q1 & Q.2 (Do this first)

Ch 21

First order Differential Equations

$$\dot{y} + ay = b \quad \text{where} \quad \dot{y} = \frac{dy_t}{dt}$$

solution $y = y_c + y_p$

y_c = complementary (general) solution

y_p = particular solution

(i) General (complementary) solution

$$\dot{y} + ay = 0 \quad \text{(homogeneous function)}$$

$$\Rightarrow \frac{\dot{y}}{y} = -a$$

LHS take an integral

$$\int \frac{\dot{y}}{y}\, dt = \int \frac{dy/dt}{y}\, dt = \int \frac{1}{y}\, dy = \ln y + c'$$

RHS take an integral

$$\int (-a)\, dt = -at + c''$$

thus $\ln y + c' = -at + c'' \Rightarrow \ln y = -at + c$ with $c^* = c' + c''$

This implies

$$y(t) = e^{-at + c^*} = \underbrace{c e^{-at}}_{\text{this is } y_c} \quad \text{with } c = e^{c^*}$$

(ii) Particular Solution

$$\dot{y} + ay = b$$

Let $\dot{y}=0$ since $\dot{y}=0$ at a steady-state level.

$$0 + a\bar{y} = b \Rightarrow \bar{y} = \frac{b}{a} = \text{this is } y_p$$

$\Rightarrow$ Solution : $y = y_c + y_p$ (sum of two solutions.)

$$\Rightarrow \boxed{y(t) = c e^{-at} + \frac{b}{a}}$$

ex 1) $\dot{y} + 2y = 8$

i) general solution

$$y_c = c e^{-at} = c e^{-2t}$$

ii) particular solution

$$y_p = \frac{b}{a} = \frac{8}{2} = 4$$

Thus

$$y_t = c e^{-2t} + 4.$$

- If the initial value is given, we can find the value of C.

if $y(0) = 2$,

$$y(0) = c e^{-2 \cdot 0} + 4 = 2 \quad \Rightarrow C = -2$$

Thus

$$y(t) = (-2) e^{-2t} + 4$$

In general,

$$\boxed{y(t) = \left(y_0 - \frac{b}{a}\right) e^{-at} + \frac{b}{a}}$$

Ex 2) $\dot{y} + 2y = 6$, $y_0 = 10$ $(a=2, b=6)$

$$y(t) = \left(10 - \frac{6}{2}\right) e^{-2t} + \frac{6}{2} = 7e^{-2t} + 3$$

Exercises

a) $\dot{y} + 4y = 0$, $y_0 = 1$.

b) $\dot{y} + y = 4$, $y_0 = 0$

c) $\dot{y} - 7y = 7$, $y_0 = 7$

Exercises H/W

Ex 21.1, p.868

Q. 1 (a), (b) & (d)

Q. 2 (a), (b)

Q. 7 (skip)

Convergence issue

$$y(t) = \left(y_0 - \frac{b}{a}\right) e^{-at} + \frac{b}{a}$$

If $a > 0$, $e^{-at} \to 0$ as $t \to \infty$

Then $y(t) \to \frac{b}{a}$

If $a < 0$, $e^{-at} \to \infty$

then $y(t)$ diverges!

eg) Capital accumulation

$$\dot{K} = \bar{I} - \delta K$$

δ = depreciation rate

I = investment.

capital accumulate if $\dot{K} > 0$. $\underset{\text{(invest)}}{I} > \underset{\text{(depreciation)}}{\delta K}$

$$\Rightarrow \dot{K} + \delta K = \bar{I}. \quad a = \delta,\ b = \bar{I}$$

$$K(t) = \left(K_0 - \frac{\bar{I}}{\delta}\right) e^{-\delta t} + \frac{\bar{I}}{\delta}$$

since $\delta > 0$, $K(t) \to \frac{\bar{I}}{\delta}$

Ch 23

Second order Differential Equations.

$$\ddot{y} + a_1\dot{y} + a_2 y = b$$

where $\ddot{y} = \frac{d^2 y_t}{dt^2}$, $\dot{y} = \frac{dy_t}{dt}$

Solution $y = y_c + y_p$

(i) General solution

$$\boxed{y_c = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$ (if 2 distinct roots)

where r_1 and r_2 are characteristic roots

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

[or they are eigenvalues of $\begin{pmatrix} a_1 & a_2 \\ 1 & 0 \end{pmatrix}$.]

why?

Suppose $y_t = ce^{rt}$. Then

$$\dot{y} = rce^{rt}, \quad \ddot{y} = r^2 ce^{rt}$$

$$\ddot{y} + a_1\dot{y} + a_2 y = r^2 ce^{rt} + a_1 rce^{rt} + a_2 e^{rt}$$

$$= (r^2 + a_1 r + a_2) ce^{rt} \overset{\text{let}}{=} 0$$ (homogeneous function)

$$\Rightarrow \quad r^2 + a_1 r + a_2 = 0.$$

Also,

$$\boxed{y_c = (c_1 + c_2 t) e^{rt}}$$ (if double roots)

$$\boxed{y_c = c_1 e^{ht}\cos vt + c_2 e^{ht}\sin vt}$$ (if complex roots)

with $h = -\frac{a_1}{2}$, $v = \frac{\sqrt{4a_2 - a_1^2}}{2}$.

Ex1) $\ddot{y} + \frac{1}{2}\dot{y} + \frac{3}{64}y = 0$

$$r^2 + \frac{1}{2}r + \frac{3}{64} = 0 \Rightarrow r = -\frac{1}{8}, -\frac{3}{8}$$

Thus

$$y(t) = c_1 e^{-\frac{1}{8}t} + c_2 e^{-\frac{3}{8}t}$$

Ex2) $\ddot{y} + \dot{y} - 2y = 0$

$$r^2 + r - 2 = 0 \Rightarrow r = 1, -2$$

$$y(t) = c_1 e^{t} + c_2 e^{-2t}$$

(ii) Particular solution

$$\ddot{y} + a_1\dot{y} + a_2 y = b$$

Let $\ddot{y} = \dot{y} = 0$, $y = \bar{y}$

$$\Rightarrow \boxed{\bar{y} = \frac{b}{a_2}}, \quad a_2 \neq 0$$

$\Rightarrow$ Thus,

$$\boxed{y = y_c + y_p = \begin{cases} c_1 e^{r_1 t} + c_2 e^{r_2 t} + \frac{b}{a_2} & (r_1 \neq r_2) \\ (c_1 + c_2 t)e^{rt} + \frac{b}{a_2} & (r_1 = r_2 = r) \\ c_1 e^{ht}\cos vt + c_2 e^{ht}\sin vt + \frac{b}{a_2} & (\text{complex roots}) \end{cases}}$$

Ex3) $\ddot{y} + \dot{y} - 2y = -10$, $y(0) = 12$, $y(0)' = -2$

$$\bar{y} = \frac{-10}{-2} = 5. \qquad r^2 + r - 2 = 0 \Rightarrow r = 1, -2$$

$$y(t) = c_1 e^{1 \cdot t} + c_2 e^{-2t} + 5 \Rightarrow 4e^{t} + 3e^{-2t} + 5$$

$$y(0) = c_1 + c_2 + 5 = 12,$$

$$y'(0) = c_1 e^{t} - 2c_2 e^{-2t} \Rightarrow y'(0) = c_1 - 2c_2 = -2$$

Thus $c_1 = 4$, $c_2 = 3$

Exercise ~~HW~~ In class H/W

a) Ex 23.2, p.918. Q1 (a), (b) also (c), (d)

b) " , p.919 Q2 (a), (b). also (c), (d)

skip Ch24.

$\boxed{\text{Ch 25}}$ Optimal Control Theory

"optimization over time"

- static optimization

  eg) profit max using capital (K).

$$\pi(K) = p \cdot q(K) - rK$$

$$\pi'(K) = p q'(K) - r = 0$$

- dynamic optimization

  eg1) Max $\int_0^T e^{-\rho t} \pi(K(t))\, dt$

  where ρ = discount rate.

  we want to choose the path of K values, $K(t)$

  eg2) Let $\dot{K} = I(t) - \delta K(t)$

$$\pi[K(t), I(t)] = p \cdot q[K(t)] - \underbrace{c}_{\text{cost of buying new investment}} \cdot I(t)$$

$$\max \int_0^T e^{-\rho t} \pi[K(t), I(t)]\, dt$$

$$\text{s.t.} \quad \dot{K} = I(t) - \delta K$$

$$K(0) = K_0$$

In general, the dynamic optimization problem is

$$\text{Max } J = \int_0^T f(x(t), y(t), t)\,dt$$

$$\text{s.t. } \quad \dot{x} = g(x(t), y(t), t)$$

$$x(0) = x_0 > 0 \text{ given}, \quad x(T) \text{ is bounded.}$$

Here, J = objective function

$x(t)$ = state variable (for which $\dot{x}$ is given)

$y(t)$ = control variable (we wish to find the path of $y(t)$)

Def Hamiltonian function H

$$H[x(t), y(t), \lambda(t), t]$$

$$= f(x(t), y(t), t] + \lambda(t)\, g[x(t), y(t), t]$$

where $\lambda(t)$ = co-state variable (as in Lagrange multiplier)

then the necessary conditions for the dynamic opt. problem are

i) $\boxed{\dfrac{\partial H}{\partial y} = 0}$; $y(t)$ is chosen to max H at each time.

ii) $\boxed{\dot{\lambda} = -\dfrac{\partial H}{\partial x}, \quad \dot{x} = g[x(t), y(t), t]}$

; $\lambda(t)$ and $x(t)$ satisfies the above equations.

iii) Two boundary conditions are

$$x(0) = x_0, \quad \boxed{\lambda(T) = 0}$$

(transversality condition)

Thus the sufficient conditions are

(i) $f(x,y,t)$ is differentiable and jointly concave in x and y.

(ii) One of the followings is true

$$\begin{cases} g(x,y,t) \text{ is linear in } (x,y) \\ g(x,y,t) \text{ is concave in } (x,y) \text{ and } \lambda(t) \geq 0. \\ g(x,y,t) \text{ is convex in } (x,y) \text{ and } \lambda(t) \leq 0 \end{cases}$$

Ex1) Max $\int_0^1 (x-y^2)\,dt$

s.t $\dot{x} = y$, $x(0) = 2$.

i) Hamiltonian function

$$H = (x-y^2) + \lambda(y)$$

solve for y $\Rightarrow$ (in terms of λ)

ii) $\dfrac{\partial H}{\partial y} = -2y + \lambda = 0 \Rightarrow y(t) = \dfrac{\lambda(t)}{2}$.

$\left(\dfrac{\partial^2 H}{\partial y^2} = -2 < 0\right.$ concave $\left.\Rightarrow \max\right)$

$\dot{x}$ is given in terms of y and thus λ $\Rightarrow$

iii) $\dot{\lambda} = -\dfrac{\partial H}{\partial x} = -1 \Rightarrow \dot{\lambda} = -1$

$$\dot{x} = y = \frac{\lambda}{2}$$

Find $\lambda(t)$ using $\lambda(T)=0$ $\Rightarrow$

iv) Boundary

$x(0) = 2$, $\lambda(1) = 0$

Now, interpret.

$\dot{\lambda} = -1 \Rightarrow \lambda(t) = c_1 - t$.

using $\lambda(1) = 0$, $\lambda(1) = c_1 - 1 \Rightarrow c_1 = 1$

thus $\lambda(t) = 1 - t$.

Then, from $\dot{x} = \frac{\lambda}{2}$,

$$\dot{x} = \frac{1-t}{2} \qquad \frac{dx(t)}{dt} = \frac{1-t}{2}$$

$$\Rightarrow x(t) = \frac{1}{2}t - \frac{1}{4}t^2 + c$$

Since $x(0) = 2$, $x(0) = \frac{1}{2} - \frac{1}{4}0^2 + c \Rightarrow c = 2$

thus

$$x(t) = \frac{1}{2} - \frac{1}{4}t^2 + 2$$

Also

$$y(t) = \frac{1}{2}\lambda(t) = \frac{1}{2}(1-t).$$

This completes the answer.

Ex2) Investment problem

$Q = K - aK^2$ where Q = output.

$\dot{K} = I - \delta K$ δ = depreciation rate

$\pi = pQ - I^2$ (if the cost of investment $= I^2$)

$= K - aK^2 - I^2$ (if $p = \$1.00$)

$$\text{Max} \int_0^T (K - aK^2 - I^2)\,dt$$

K = state variable

I = control "

s.t. $\dot{K} = I - \delta K$

$K(0) = K_0$ given.

i) Hamiltonian

$$H = (K - aK^2 - I^2) + \lambda(I - \delta K)$$

ii) $\frac{\partial H}{\partial I} = -2I + \lambda = 0 \Rightarrow I(t) = \frac{1}{2}\lambda(t)$

(since $\frac{\partial^2 H}{\partial I^2} = -2 < 0$, it gives a max.)

iii) $\dot{\lambda} = -\frac{\partial H}{\partial K} = -1 + 2aK + \lambda\delta$

$\dot{K} = I - \delta K = \frac{\lambda}{2} - \delta K$

iv) $\lambda(T) = 0 \quad , \quad K(0) = K_0$

Now, we need to solve the system of differential equations

(ch 24)

$$\begin{bmatrix} \dot{\lambda} \\ \dot{K} \end{bmatrix} = \begin{pmatrix} \delta & 2a \\ \frac{1}{2} & -\delta \end{pmatrix} \begin{pmatrix} \lambda \\ K \end{pmatrix}$$

Note
$\bar{\lambda}$ & $\bar{K}$ are given from (iii) by setting $\dot{\lambda}=0$ & $\dot{K}=0$ then jointly solve
$\bar{\lambda} = \frac{\delta}{\delta^2+a}$, $\bar{K} = \frac{1}{2(\delta^2+a)}$

$$\Rightarrow \begin{cases} \lambda(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \bar{\lambda} \\ K(t) = \frac{r_1 - \delta}{2a} c_1 e^{r_1 t} + \frac{r_2 - \delta}{2a} c_2 e^{r_2 t} + \bar{K} \end{cases}$$

$\Rightarrow I(t) = \frac{1}{2}\lambda(t)$

where r_1 and r_2 are eigenvalues of

$$\begin{pmatrix} \delta & 2a \\ \frac{1}{2} & -\delta \end{pmatrix}$$

where $r_1, r_2 = \pm\sqrt{\delta^2 + a}$

Thus, using $K(0)=0$, $\lambda(T)=0$, we can find c_1 & c_2 (see text 1011)

Note $\lambda(t)$ = marginal (imputed) value of the state vari
= shadow price of state var, $x(t)$

Exercises Ex 25.1 p.1012

Q.1, Q.2 and Q.3

Q.1: in class

Q.2: system

Q.3: not autonomous ~~nonhomogeneous~~ (skip)

Optimization Problems Involving Discounting

$$\max\ J = \int_0^T F(x(t), y(t))\, e^{-\rho t}\, dt$$

$$\text{s.t.}\quad \dot{x} = G[x(t), y(t)]$$

$$x(0) = x_0$$

- Hamiltonian function

$$H = F(x, y)\, e^{-\rho t} + \lambda G(x, y)$$

- $\frac{\partial H}{\partial y}$

$$\frac{\partial H}{\partial y} = F_Y(x, y)\, e^{-\rho t} + \lambda G_Y(x, y) = 0$$

$$\Rightarrow F_Y(x, y) + \lambda e^{\rho t} G_Y(x, y) = 0$$

- $\dot{\lambda}$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -e^{-\rho t} F_x(x, y) - \lambda G_x(x, y) \qquad \text{— (1)}$$

multiply $e^{\rho t}$

$$\dot{\lambda} e^{\rho t} = -F_x(x, y) - \lambda e^{\rho t} G_x(x, y) \qquad \text{— (*)}$$

where y is a function of x and $\lambda e^{\rho t}$

say, $y = \phi(x, \lambda e^{\rho t})$.

- $\dot{x}$

$$\dot{x} = G(x, \phi(x, \lambda e^{\rho t})) \qquad \text{— (2)}$$

- Boundary conditions

$$x(0) = x_0, \quad \lambda(T) = 0$$

<u>Note</u> Let $\mu(t) = \lambda(t) e^{\rho t}$

$$\dot{\mu} = \rho \lambda e^{\rho t} + \dot{\lambda} e^{\rho t} = \rho \mu + \dot{\lambda} e^{\rho t}$$

Then, (*) becomes

$$\dot{\mu} = \rho \mu - F_x(x, \underbrace{\phi(x,\mu)}_{y}) - \mu G_x(x, \underbrace{\phi(x,\mu)}_{y}) \quad \cdots (3)$$

Then we have a system of differential equation in two variables, μ and x: (2) and (3).

<u>point</u>
- (3) is autonomous (not depending on t)
- (1) is not autonomous (depending on t via $e^{-\rho t}$)

<u>Def</u> The current-valued Hamiltonian, $\mathcal{H}$ is defined as

$$\mathcal{H}[x(t), y(t), \mu(t)] = F(x(t), y(t)) + \mu G[x(t), y(t)]$$

where $\mu = \lambda e^{\rho t}$ (λ is transformed into μ!)

$$\mathcal{H} = H e^{\rho t}$$

Necessary conditions are

(i) $\frac{\partial \mathcal{H}}{\partial y} = 0$

(ii) $\dot{\mu} - \rho \mu = - \frac{\partial \mathcal{H}}{\partial x}$ (co-state variable is μ!)

(iii) $\dot{x} = g(x(t), y(t))$

(iv) Boundary conditions

$$x(0) = x_0, \quad \mu(T) e^{-\rho T} = 0$$

why?

$$\mathcal{H} = He^{\rho t} \Rightarrow H = \mathcal{H} e^{-\rho t}$$

(ii)

- $\frac{\partial H}{\partial y} = 0$ implies $\frac{\partial \mathcal{H}}{\partial y} e^{-\rho t} = 0 \Rightarrow \frac{\partial \mathcal{H}}{\partial y} = 0$ since $e^{-\rho t} \neq 0$

- $-\frac{\partial H}{\partial x} = -\frac{\partial \mathcal{H}}{\partial x} e^{-\rho t}$

$$\dot{\lambda} = -\rho \mu e^{-\rho t} + \dot{\mu} e^{-\rho t} \quad (\text{since } \lambda = \mu e^{-\rho t})$$

since $\dot{\lambda} = -\frac{\partial H}{\partial x}$

$$-\frac{\partial \mathcal{H}}{\partial x} \cancel{e^{-\rho t}} = -\rho\mu \cancel{e^{-\rho t}} + \dot{\mu} \cancel{e^{-\rho t}}$$

$$-\frac{\partial \mathcal{H}}{\partial x} = \dot{\mu} - \rho\mu \quad \cdots \text{(iii)}$$

- $\lambda(T) = 0 \Rightarrow \mu(T) e^{-\rho T} = 0.$

Ex1) $\text{Max} \int_0^T e^{-\rho t}(ax - bx^2 - cy^2)\,dt$

s.t. $\dot{x} = y - \alpha x$

$x(0) = x_0$

i) Current-valued Hamiltonian function

$$\mathcal{H} = ax - bx^2 - cy^2 + \mu(y - \alpha x)$$

ii) $\frac{\partial \mathcal{H}}{\partial y} = -2cy + \mu = 0 \Rightarrow y = \mu/2c$

iii) $\dot{\mu} - \rho\mu = -\frac{\partial \mathcal{H}}{\partial x} = -(a - 2bx - \mu\alpha)$

$$\dot{\mu} = (\rho + \alpha)\mu + 2bx - a$$

$$\dot{x} = \mu/2c - \alpha x$$

$$\Rightarrow \mu(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \bar{\mu}$$

$$x(t) = \frac{r_1 - \rho - \alpha}{2b} C_1 e^{r_1 t} + \frac{r_2 - \rho - \alpha}{2b} C_2 e^{r_2 t} + \bar{x}$$

where r_1 & r_2 are eigenvalues of

$$\begin{pmatrix} \rho + \alpha & 2b \\ M/2C & -\alpha \end{pmatrix}$$

Ex2) Invest problem with discounting

$$\text{Max} \int_0^T e^{-\rho t} [f(k) - I^2] \, dt$$

$$\text{s.t} \quad \dot{k} = I - \delta k$$

$$k(0) = k_0$$

i) Current-valued Hamiltonian function

$$H = f(k) - I^2 + \mu (I - \delta k)$$

ii) $\dfrac{\partial H}{\partial I} = -2I + \mu = 0 \quad \Rightarrow I = \mu/2$

$\left(\dfrac{\partial^2 H}{\partial I^2} = -2 ; \text{ concave} \Rightarrow \max \right)$

iii) $\dot{\mu} - \rho\mu = -\dfrac{\partial H}{\partial k} = -f'(k) + \mu\delta$

$$(*) \quad \left[\begin{array}{l} \dot{\mu} = (\rho + \delta)\mu - f'(k) \\ \dot{k} = \frac{1}{2}\mu - \delta k \end{array} \right.$$

Due to $f'(k)$, we cannot find an explicit solution.
thus, we can perform a qualitative analysis.

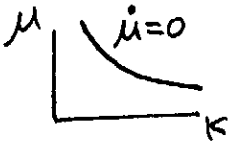
Qualitative analysis

(i) $\dot{\mu} = 0 \Rightarrow (\rho+\delta)\,\mu = f'(k)$

$$\Rightarrow \mu = \frac{f'(k)}{\delta+\rho}$$

$$\frac{d\mu}{dk} = \frac{1}{\delta+\rho} f''(k) < 0 \quad \text{since } f''(k) < 0$$

Assume $f'(0) = \infty$, $f'(\infty) = 0$



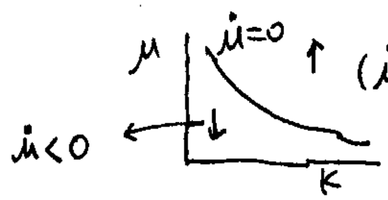
$\Leftarrow$ ($\frac{d\mu}{dk} < 0 \Rightarrow$ negatively sloped

$f'(0) = \infty \Rightarrow$ never touches μ axis

$f'(\infty) = 0 \Rightarrow$ " k axis

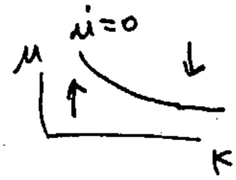
(ii) $\frac{\partial \dot{\mu}}{\partial \mu} = \rho + \delta > 0$ from $\dot{\mu} = (\rho+\delta)\mu - f'(k)$

$\Rightarrow$ An increase in μ leads to an increase in $\dot{\mu}$.



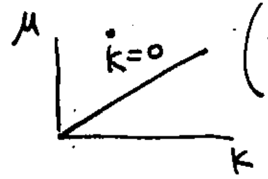
(vertical movement as μ changes)

Note If $\frac{\partial \dot{\mu}}{\partial \mu} < 0$,



(iii) $\dot{k} = 0 \Rightarrow \frac{1}{2}\mu = \delta k$ or $\mu = 2\delta k$

$\frac{d\mu}{dk} = 2\delta > 0$. Thus



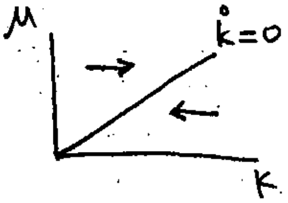
(slope $= 2\delta$

positively sloped

(linear line)

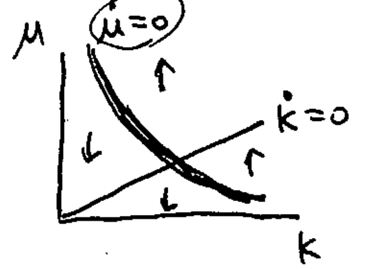
(iv) $\frac{\partial \dot{k}}{\partial k} = -\delta < 0$ from $\dot{k} = \frac{1}{2}\mu - \delta k$

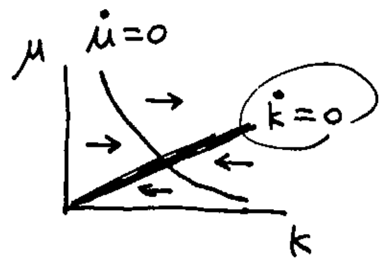
$\Rightarrow \dot{k}$ is negative to the right, and positive to the left of the $\dot{k} = 0$. (As k increases, $\dot{k}$ falls)



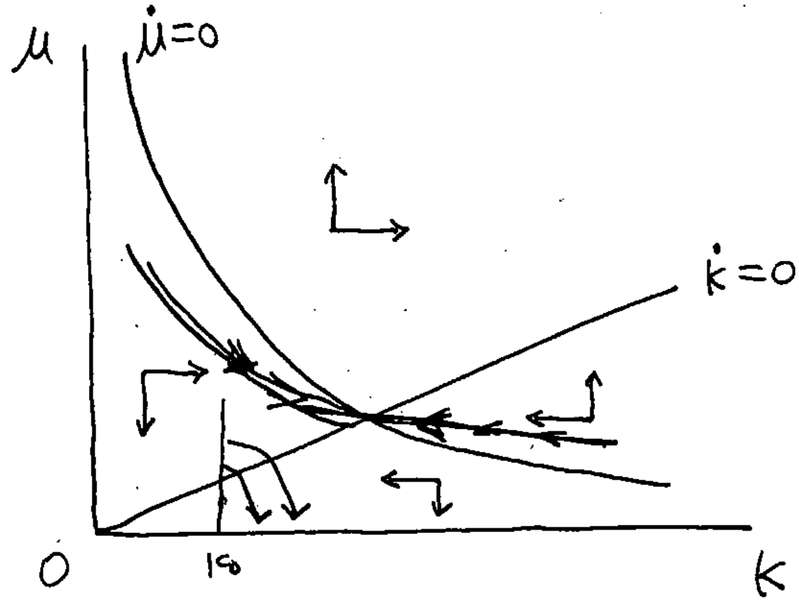
(horizontal movement as k changes)

That is,





Combining the results together, we have the phase diagram



How about the equilibrium?

$\Rightarrow$ the steady state is a saddle-point equil.

why?

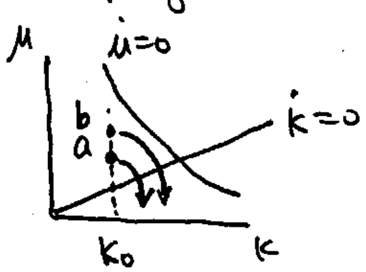
i) check definiteness from (*), p.22

$$H = \begin{bmatrix} \rho+\delta & -f'' \\ \frac{1}{2} & -\delta \end{bmatrix} \qquad |H_1| = \rho+\delta > 0$$

$$|H_2| = \underbrace{-\delta(\rho+\delta)}_{-} + \frac{1}{2}\underbrace{f''}_{-} < 0$$

thus, neither p.d. or n.d.
Implying that eigenvalues are of opposite sign.

ii)



starting at $k=k_0$, at the point a, the path of the trajectory moves to $\mu(T)=0$.

$\Rightarrow$ Investment follows the same pattern ($I=\mu/2$)

Exercises Ex 25.2, p.1024

In class Q.1, Q.2 (skip Q.3) * use Q.2 in-class exercise
Q.6&7 qualitative analysis

Alternative Boundary Conditions on $x(T)$

i) fixed endpoint

$$x(T) = b$$

.. Replace the transversality condition $\mu(T) = 0$ with $x(T) = b$. Thus, we have

$$x(0) = x_0, \quad x(T) = b.$$

ii) Inequality-constrained endpoint

$$x(T) \geqslant b$$

Ex) Optimal depletion of an exhaustible resource

"Suppose that you live alone on a desert island. You live from time 0 (now) to T. You have a fixed stock of an exhaustible resource which is not reproducable. You need to choose a consumption path."

consumption ↓ control variable

$$\max \int_0^T e^{-\rho t} U[c(t)]\,dt$$

s.t. $\dot{R} = -c$ ... $R(t)$ declines by the amount consumed $c(t)$.

$$R(0) = R_0$$

$$R(T) \geqslant 0$$

where U is the utility function, $U' > 0$, $U'' < 0$.

· The current-valued Hamiltonian functn is

$$\mathcal{H} = U(c(t)) - \mu c$$

· $\frac{\partial \mathcal{H}}{\partial c} = u'(c) - \mu = 0 \Rightarrow c = \phi(\mu)$ .. we wish to have a solution in terms of a control variable : $c(t)$, here.

$\left(\frac{\partial^2 \mathcal{H}}{\partial c^2} = u'' < 0 \right.$. concave ; max)

· $\dot{\mu} - \rho\mu = -\frac{\partial \mathcal{H}}{\partial R} = 0$

Thus

$$\dot{\mu} = \rho \mu \rightarrow \mu(t) = c_1 e^{\rho t}$$

$$\dot{R} = -c = -\phi(\mu) \rightarrow R(t) = -\int_0^t \phi(c_1 e^{\rho t}) ds + c$$

since $R(0) = R_0$, we have $c' = R_0$

Evaluating at $t = T$, and $R(T) = 0$

$$R(t) = R_0 - \int_0^T \phi(c_1 e^{\rho s}) ds = 0.$$

Exercise EX 25.3, p. 1039

Q.1 , Q.2
(skip) (skip)

Infinite-Time Horizon problem

$$\text{Max } J = \int_0^\infty F(x,y)e^{-\rho t}\,dt$$

$$\text{s.t. } \dot{x} = G(x,y)$$

$$x(0) = x_0, \quad \lim_{t\to\infty} x(t) = b$$

: Most results carry over.

Ex) Neoclassical Growth Model

$$\text{Max } \int_0^\infty e^{-(\rho-n)t}\,U[c(t)]\,dt$$

$$\text{s.t. } \dot{k} = f(k) - c - (\delta+n)k$$

$$k(0) = k_0 > 0$$

$$k(t) \geq 0$$

$$c(t) \geq 0$$

where n = population growth rate

· Current valued Hamiltonian

$$H = U(c) + \mu\,(f(k) - c - (\delta+n)k)$$

· $\dfrac{\partial H}{\partial c} = U'(c) - \mu = 0 \quad \Rightarrow c = \phi(\mu)$

· $\dot{\mu} - (\rho-n)\mu = -\dfrac{\partial H}{\partial k} = -\mu\,[f'(k) - (\delta+n)]$

thus

$$\left[\begin{array}{l} \dot{\mu} = \mu\,[\rho+\delta - f'(k)] \\ \dot{k} = f(k) - \phi(\mu) - (\delta+n)k \end{array}\right.$$

· Qualitative analysis

<u>Note</u> we wish to have the analysis in the (c, k) diagram rather than the (μ, k) diagram

From $u'(c) - \mu = 0 \qquad \left(\frac{\partial H}{\partial c} = 0\right)$

$$u''(c)\,\dot{c} = \dot{\mu}$$

$$\Rightarrow \dot{c} = \frac{\dot{\mu}}{u''(c)}$$

<u>Note</u>

$\mu = u'(c)$

from $\frac{\partial H}{\partial c} = 0$

$\dot{\mu} = \mu(\rho + \delta - f'(k))$

then

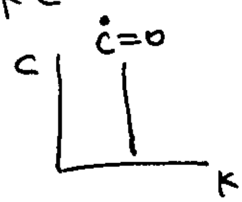
$$\dot{c} = \frac{u'(c)}{u''(c)}\,[\rho + \delta - f'(k)]$$

Then two systems are

$\dot{c}$ and $\dot{k}$.

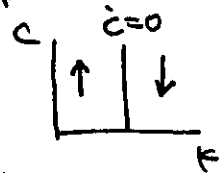
i) $\dot{c} = 0$

$\rho + \delta = f'(k)$ ... not a function of c



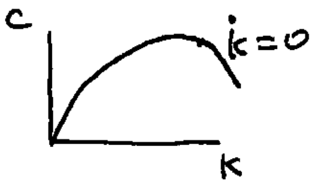
ii) $\dfrac{\partial \dot{c}}{\partial k} = -\dfrac{\overset{>0}{u'(c)}}{\underset{<0}{u''(c)}}\, \overset{<0}{f''(k)} < 0$

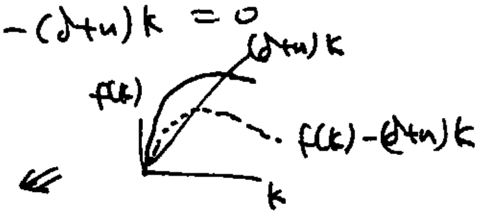
: this implies that c falls at the right side of $\dot{c} = 0$
(increase) (left)
(movement of c vertically)



iii) $\dot{k} = 0$: $\quad \dot{k} = f(k) - \phi(\mu) - (\delta + n)k = 0$

$$c = f(k) - (\delta + n)k$$





slope $= \dfrac{dc}{dk} = f'(k) - (\delta + n)$

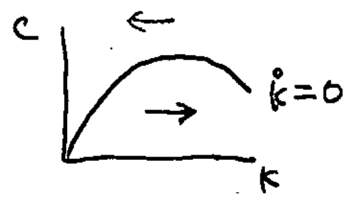
> 0 if $f'(k) > (\delta + n)$

< 0 if $f'(k) < (\delta + n)$

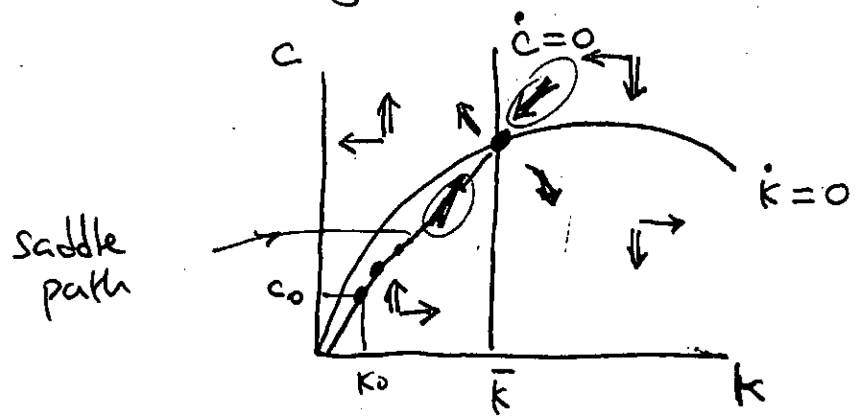
$= 0$ if $f'(k) = \delta + n$

iv) $\frac{\partial \dot{k}}{\partial c} = -1 < 0$ (movement of k horizontally)

This implies that k decreases above the $\dot{k}=0$ curve
(increases) (below)



Combining together



The steady-state equilibrium is a saddlepoint.

why? The coef. matrix from
$$\begin{cases} \dot{c} = \frac{u'}{u''}(\rho+\delta-f'(k)) \\ \dot{k} = f(k) - c - (\delta+n)k \end{cases}$$

$$H = \begin{bmatrix} \frac{\partial}{\partial c}\left(\frac{u'}{u''}\right)\overbrace{(\rho+\delta-f'(k))}^{0}, & -\left(\frac{u'}{u''}\right)f''(k) \\ -1, & f'(k)-(\delta+n) \end{bmatrix}$$

$|H_1| = 0$, $|H_1'| = 0$, or $\underbrace{f'(k)-(\delta+n)}_{\gtrless 0}$ any sign

$$|H_2| = |H| = -\underbrace{\frac{u'}{u''}}_{<0}\overbrace{f''(k)}^{<0} < 0.$$

The matrix is indefinite (eigenvalues are of opposite sign.)

The opt. trajectory must start on the saddle point with $k(0) = k_0$. Then it moves along the saddle point, and reaches the steady equil

<u>Exercise</u> Ex 25.4, p. 1052 (H/W)

Q.1. $c(t)$ path

Q.3. (k, c) space

More Results (Lecture 9)

1. Simultaneous system of differential equations
2. phase diagram of nonlinear systems.

Simultaneous Systems of Differential Equations

Ch24.4, p.946

Ex) $\dot{y}_1 = y_1 - 3y_2$

$\dot{y}_2 = \frac{1}{4}y_1 + 3y_2$] want to find the joint solution.

Note $\dot{y}_1 = a y_1 \Rightarrow y_1(t) = c e^{-at}$; one equation.

Rewrite the system

$$\begin{bmatrix} 1 & -3 \\ \frac{1}{4} & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix}$$

or $A \quad Y \quad = \dot{y}$

Rule Find the eigenvalues of $A \Rightarrow r_1, r_2$ (distinct roots)

Find the eigenvectors of $A \Rightarrow g_1, g_2$

then, the solution is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = c_1 g_1 e^{r_1 t} + c_2 g_2 e^{r_2 t}$$

eigenvalues of $A = \frac{3}{2}, \frac{5}{2}$

eigenvectors of $A = \begin{pmatrix} 1 \\ -1/6 \end{pmatrix}, \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$

thus

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -\frac{1}{6} \end{pmatrix} e^{\frac{3}{2}t} + c_2 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{\frac{5}{2}t}$$

ie. $y_1(t) = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{5}{2}t}$

$y_2(t) = -\frac{1}{6} c_1 e^{\frac{3}{2}t} - \frac{1}{2} c_2 e^{\frac{5}{2}t}$

Exercise $\dot{y}_1 = 4y_1 - y_2$, $\dot{y}_2 = -4y_1 + 4y_2$. Find the solution.

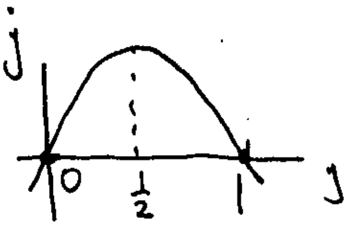
Phase diagram Analysis

(qualitative analysis)

Ex1) $\dot{y} = y - y^2$: ch22.1, p.880

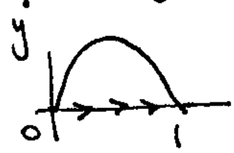
i) $\dot{y} = 0 \Rightarrow y - y^2 = y(1-y) = 0$. $y = 0$ or 1.

($\dot{y}' = d\dot{y}/dy = 1 - 2y = 0$. $\dot{y}'' = -2 < 0$)
Thus max at $y = \frac{1}{2}$



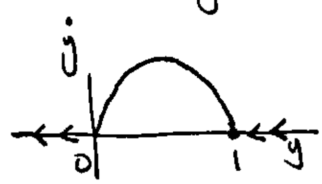
ii) $\dot{y} > 0$ if $y(1-y) > 0$. ie if $0 < y < 1$

that is, y increases if $0 < y < 1$

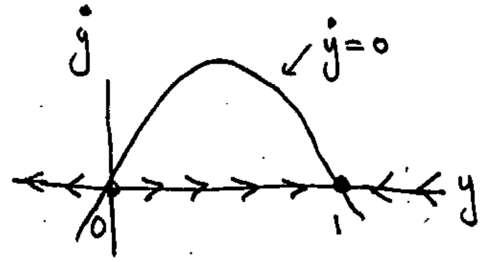


iii) $\dot{y} < 0$ if $y(1-y) < 0$ ie if $y < 0$ or $y > 1$

That is, y decreases if $y < 0$ or $y > 1$



Combining these two graphs, we have the arrows of motion



thus, $y(t)$ converges to the value $y = 1$.
($y = 0$ is not a stable solution)

Exercise $\dot{y} = 3y^2 - 2y$.

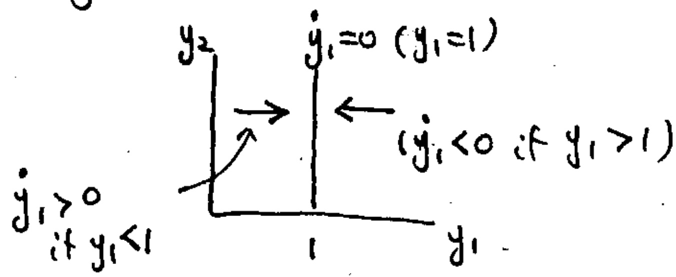
Ex 2) $\dot{y}_1 = -2y_1 + 2$ Draw the phase diagram.

$\dot{y}_2 = -3y_2 + 6$

i) $\dot{y}_1 = 0$ if $-2y_1 + 2 = 0$. ie $y_1 = 1$ $\Rightarrow$ $y_1 = 1$ is "isocline".

$\dot{y}_1 > 0$ if $-2y_1 + 2 > 0$. ie $y_1 < 1$ $\Rightarrow$ y_1 increases if $y_1 < 1$

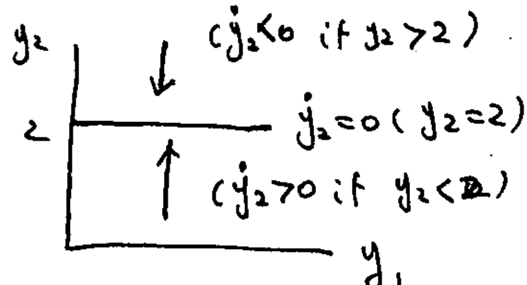
$\dot{y}_1 < 0$ if $-2y_1 + 2 < 0$. ie $y_1 > 1$ $\Rightarrow$ y_1 decreases if $y_1 > 1$



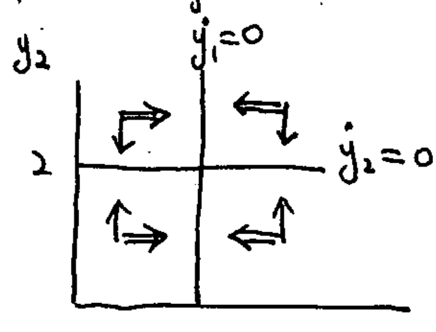
ii) $\dot{y}_2 = 0$ if $-3y_2 + 6 = 0$. ie $y_2 = 2$ $\Rightarrow$ $y_2 = 2$ is "isocline".

$\dot{y}_2 > 0$ if $-3y_2 + 6 > 0$. ie $y_2 < 2$ $\Rightarrow$ y_2 increases if $y_2 < 2$

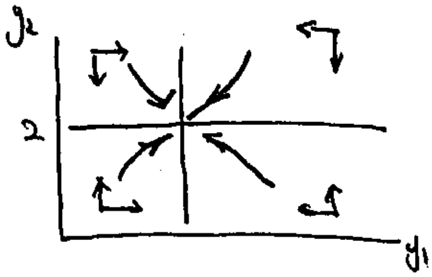
$\dot{y}_2 < 0$ if $-3y_2 + 6 < 0$. ie $y_2 > 2$ $\Rightarrow$ y_2 decreases if $y_2 > 2$



Putting them together



Thus



The trajectory is the path of y_1 & y_2

$\Rightarrow$ stable solution

Ex 3) $\dot{y}_1 = y_2 - 2$

$\dot{y}_2 = \frac{1}{4} y_1 - \frac{1}{2}$

or $\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}$

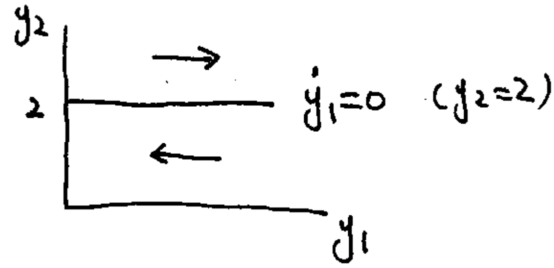
$(\dot{y} = Ay + b)$

Note
Eigenvalues of $A = -\frac{1}{2}, \frac{1}{2}$ (sign alternates $\Rightarrow$ saddle-point equilibrium)

i) $\dot{y}_1 = 0 \Rightarrow y_2 = 2$

$\dot{y}_1 > 0$ if $y_2 > 2$; y_1 increases if $y_2 > 2$

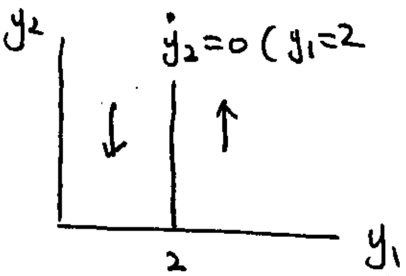
$\dot{y}_1 < 0$ if $y_2 < 2$: y_1 decreases if $y_2 < 2$



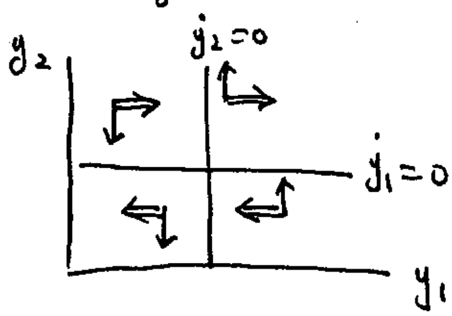
ii) $\dot{y}_2 = 0 \Rightarrow y_1 = 2$

$\dot{y}_2 > 0$ if $y_1 > 2$: y_2 increases if $y_1 > 2$

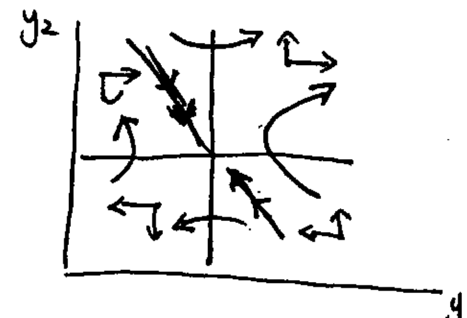
$\dot{y}_2 < 0$ if $y_1 < 2$: y_2 increases if $y_1 < 2$



Putting them together



thus,



saddle-point equil.
(partially stable)

Ex 4) $\dot{y}_1 = a y_1 - b y_2^{b-1}$ p.972

$\dot{y}_2 = y_1 - c y_2$ (nonlinear phase diagram)

$a, c > 0$, $0 < b < 1$.

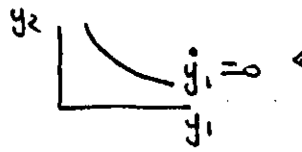
$$A = \begin{bmatrix} F_1^1 & F_1^1 \\ F_2^2 & F_2^2 \end{bmatrix} = \begin{bmatrix} a & -b(b-1)y_2^{b-2} \\ 1 & -c \end{bmatrix}$$

<u>Note</u> If $|A| < 0$, ie $|A_2| = |A| = -ac + b(b-1)y_2^{b-2} < 0$

then, the sign of eigenvalues alternates. Thus, we have a <u>saddle point</u> equilibrium (see p.964, Table).

i) $\dot{y}_1 = 0 \Rightarrow y_2 = \left(\frac{a}{b} y_1\right)^{1/(b-1)}$: y_1 isocline.

Here, as $y_1 \to 0$, $y_2 \to \infty$. As $y_1 \to \infty$, $y_2 \to 0$

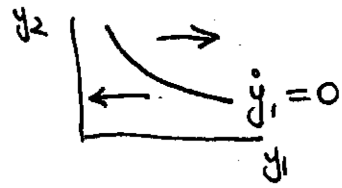


$\leftarrow y_2 = \left(\frac{a}{b} y_1\right)^{1/(b-1)}$

$\left(\text{slope} = \frac{dy_2}{dy_1} = \frac{1}{b-1}\left(\frac{a}{b} y_1\right)^{\frac{1}{b-1}-1} < 0 \right)$

ii) $\frac{\partial \dot{y}_1}{\partial y_1} = a > 0$

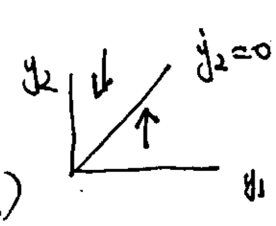
∴ y_1 increases at the points to the right of the isocline.



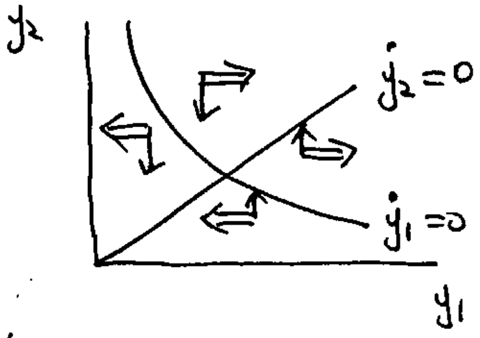
iii) $\dot{y}_2 = 0 \Rightarrow y_1 - c y_2 = 0 \Rightarrow y_2 = \frac{y_1}{c}$

iv) $\dot{y}_2 > 0$ if $y_1 - c y_2 > 0$, ie if $y_2 < \frac{1}{c} y_1$

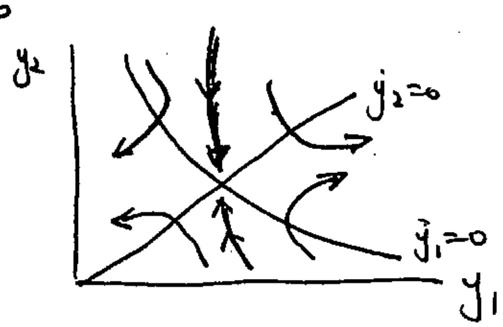
$\dot{y}_2 < 0$ if $y_2 > \frac{1}{c} y_1$ (above the y_2 isocline.)



Putting them together



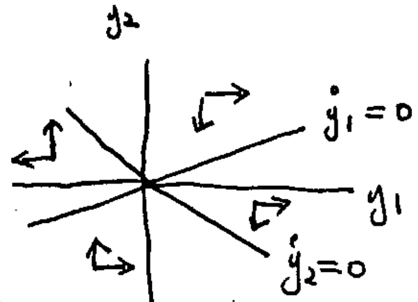
thus



saddle point equil.

<u>Exercise</u> Q.2 (p.975, Ex 24.2)

$$\dot{y}_1 = 2y_1 - 9y_2$$
$$\dot{y}_2 = -3y_1 - 4y_2$$

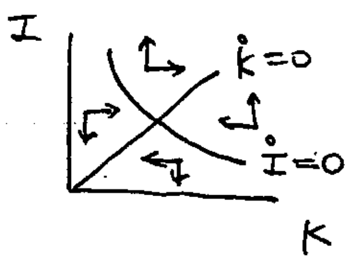


<u>Exercise</u> Q.5 (p.996, Review question, ch 24)

$$\dot{I} = \delta I - \frac{\alpha K^{\alpha-1}}{2}$$
$$\dot{K} = I - \delta K$$

<u>Note</u>

$$A = \begin{bmatrix} \delta & \frac{-\alpha(\alpha-1)}{2} K^{\alpha-2} \\ 1 & -\delta \end{bmatrix}$$



$|A| < 0$