Advanced Insurance Economics

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Problems:

1. There are four states of the weather. Every day is either hot or cold. It is also either dry or wet. Thus the four states: {HD, CD, HW, CW}. You live underground in a room with no windows, but you have a thermometer, which tells you if it is hot or cold outside. However, you have no way of telling if it is dry or wet. You keep data and have estimated the probabilities of hot and cold.

(a) Describe the probability space $(\Omega, \Im, p)$.

(b) How would your answer change if you also had a hygrometer, which can tell if it is dry or wet?

(c) Suppose your friend had the hygrometer. She also keeps records and has estimated probabilities of dry and wet. If you combine her information with your information, will you arrive at the same probability space as in part (b)? Why or why not?

2. Consider distribution functions with supports contained in (a,b). Define the following preferences functionals:

(i) $V(F) \equiv \mu_F$ (the mean of F)
(ii) $V(F) \equiv$ median of F
(iii) $V(F) \equiv$ max Supp (F)
(iv) $V(F) \equiv$ min Supp (F)
(v) $V(F) \equiv f(\mu_F, \sigma_F^2)$ (a function of mean and variance)

For preference functionals (i)-(iv) above, find examples of simple lotteries to convince yourself that these preferences are unrealistic. Do you think preference functional (v) is realistic? Why or why not?

3. Find the degrees of absolute and relative risk aversion for each utility function below.

(a) $u(y) = -e^{-ry}$, $r > 0$
(b) $u(y) = \ln y$
(c) $u(y) = \frac{1}{1-\gamma} y^{1-\gamma}$
(d) $u(y) = y - ky^2$, $k > 0$, $y < \frac{1}{2k}$
(e) $u(y) = \frac{1}{b-1}(a+by)^{1-(1/b)}$, $a + by > 0$, $b \notin \{0, -1\}$

4. For $u(y) = y - ky^2$, show that the EU preference functional has form (v) in question 1.

5. A consumer has \$100 in wealth plus a lottery ticket. The lottery ticket pays a prize of \$100 with probability p = ½. Otherwise, the prize is zero. The consumer has von Neumann-Morgenstern utility exhibiting constant relative risk aversion, with the level of relative risk aversion $\gamma = 1$.

(a) What is the lowest price that consumer would accept to sell this lottery ticket?

(b) If the consumer had \$100, but did not own the lottery ticket, how much would she be willing to pay to buy the ticket?

6. Assume $u''' > 0$ and define absolute prudence as $p(y) \equiv \frac{-u'''(y)}{u''(y)}$.

Now define $v(y) \equiv -u'(y)$ and note that $v(y)$ has the properties of a risk-averse utility function. Show the equivalence of the following three statements:

(a) u exhibits DARA
(b) $p(y) > r(y) \forall y$
(c) $v(y)$ is more risk averse than $u(y)$.

7. Let $\tilde{Y} = W - \tilde{x}$, with insurance available for the loss $\tilde{x}$ and with $\lambda > 0$. Let α^* denote the optimal coinsurance level.

(a) Suppose that preferences satisfy CARA. Show that α^* remains unchanged for $\tilde{Y} = kw - \tilde{x}, k > 0$.
(b) Suppose that preferences satisfy CRRA. Show that α^* remains unchanged for $\tilde{Y} = k\tilde{w} - k\tilde{x}, k > 0$.

8. Consider a state-claims model with two states of nature. What is the slope of the price line for insurance contracts? Use this information to demonstrate Mossin's Theorem.

9. Consider a model of deductible insurance. Show that Mossin's Theorem also holds for deductibles:

(1) $\lambda = 0 \Rightarrow$ Full insurance is optimal, $D^* = 0$.
(2) $\lambda > 0 \Rightarrow$ Partial coverage is optimal, $D^* > 0$.

10. Set up a model of insurance coverage for a policy with an upper limit. The indemnity payment is specified is $I(x) = \min(x, \theta)\theta$ is the upper limit, chosen by the consumer. Assume the premium is set as $P(\theta) = (1 + \lambda)E[I(\tilde{x})]$. What is the first-order condition for the optimal choice of an upper limit?
(Remark: I did not ask about Mossin's Theorem, which still holds, but is much more difficult here, since EU is not differentiable at full coverage!)

11. Let $u(y) \equiv y - ky^2, k > 0, y < 1/2k$. Let $\tilde{\varepsilon}$ be an independent background risk, $E\tilde{\varepsilon} = 0$. Show that the optimal level of insurance α^* is the same both with and without the background risk. (Note that $u''' = 0$ for quadratic utility.)

12. A consumer has a probability *p* of incurring a loss of size *L*, but can buy a coinsurance contract which pays αL in case of a covered loss. Unfortunately, there are many exceptions in the contract. If there is a loss, there is only a probability *q* that the loss is covered, $q < 1$. With a probability (1-*q*) the loss will not be covered. However, in this case the insurer will refund the premium *P*. (Note: this last feature makes this insurance policy different than one with "default risk," as we studied in class.) The insurance premium *P* is actuarially fair (i.e. earns an expected profit of zero for the insurer).

(a) Write the fair premium *P* as a function of α, *p*, *q* and *L*. (Be sure to remember that this premium is sometimes refunded.)

(b) Describe the first-order and second-order conditions for maximizing Didi's expected utility.

(c) With no default risk, Mossin's Theorem tells us that $\alpha^* = 1$. But with default risk, we saw in class that $\alpha^* < 1$. But this problem is different than both of these cases. Determine whether the optimal level of insurance is full coverage ($\alpha^* = 1$) or less than full coverage ($\alpha^* < 1$).

13. Consider this simplified savings problem. A consumer receives an income of *w* in each of two periods. The consumer can save an amount s in the first period, and delay consumption until the second period. We assume an interest rate of zero and no impatience by the consumer in delaying her consumption. Thus, "lifetime" utility can be given as a function of saving: $U(s) \equiv u(w-s) + u(w+s)$. We assume that *u* is concave and that u' is convex. We allow *s* to be negative (which we interpret as borrowing money).

(a) Find the optimal level of savings, *s**. (Check both first and second-order conditions.)

(b) Suppose we introduce a small zero-mean background risk $\tilde{\varepsilon}_i$ at date *i*. We assume that $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ are identical and statistically independent of one another. Will *s** rise or fall? Explain. (If the answer depends on more information, what else do you need to know?)

(c) Suppose we introduce the background risk $\tilde{\varepsilon}_i$ in one period, but not the other. Will *s** rise or fall? Explain.

(d) Suppose instead that s is mandatory, imposed by the government at s = (0.1)w. The consumer has background risk in both periods, $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$, but the government will eliminate it in period 1 or in period 2 as a public service. Would the consumer choose eliminating the risk $\tilde{\varepsilon}_1$ or the risk $\tilde{\varepsilon}_2$? Explain.

14. Both $\tilde{x}$ and $\tilde{y}$ have supports contained in the open interval (a,b). Show that $E\tilde{x} = E\tilde{y}$ if only if $\int_a^b F(x)dx = \int_a^b G(x)dx.$

15. Suppose that $F \sim Unif[a,b]$ and that $G \sim Unif[A,B]$. Give conditions on {a,b,A,B} such that $F \text{ FSD } G$. Give conditions such that $F \text{ SSD } G$.

16. Let $\tilde{X}$ denote an equally weighed lottery with prizes 5 and 10. Let $\tilde{Y}$ denote an equally weighted lottery with prizes 0 and 15.

(a) Explain how $\tilde{Y}$ is a simple mean-preserving spread of $\tilde{X}$.

(b) Use the single crossing property of the distribution functions to explain why $\tilde{Y}$ is riskier than $\tilde{X}$ in the sense of Rothschild and Stiglitz.

(b) Find $\tilde{\varepsilon}$ such that $E(\tilde{\varepsilon} | \tilde{x} = k)$ x) $\forall k$ and $\tilde{Y} =_d \tilde{X} + \tilde{\varepsilon}$.

17. A risk-averse art collector owns two original Picasso paintings (painted by Diego Picasso, no relation to Pablo). Each painting is worth € 40 and each has a probability $p = 1/4$ of being stolen. Because the paintings are kept at different locations, these risks are independent from one another. Deductible insurance is available at a fair price, but the level of insurance is limited to a total insurance premium of € 10.

(a) Suppose that the collector must purchase a separate policy for each painting, with deductible levels D_1 and D_2 respectively. Show that spending an equal insurance premium of € 5 per painting is optimal.

(b) Suppose that the collector is allowed to use her € 10 to purchase one policy for both paintings, with an aggregate deductible D. Determine the level of D.

(c) Show whether or not this aggregate deductible is preferred to the two separate deductibles.

18. Let $\tilde{x}$ be a random variable that has a payoff of $(-p_2)$ with probability p_1, and a payoff of $(+p_1)$ with probability p_2. Note that $E\tilde{x} = 0$. Consider $t\tilde{x}, t \geq 0$. Define $k(t)$ via $\mu \sim \mu + t\tilde{x} + k(t)$. Suppose that risk aversion is of order one, so that $\lim_{t \to 0^+} k'(t) > 0$.

Show that the slope of an indifference curve in state-claims space is steeper than p_1 / p_2, even as t approaches zero.

19. Yaari's "Dual Theory" models a preference functional for choice under risk as:

$$V(F) \equiv \int y d[g(F(y))],$$

where F is the cumulative distribution function and $g:[0,1] \to [0,1]$ is an increasing function, and g is strictly concave if there is risk aversion. Consider a two-state world with probability p_i for state i =1,2. What do the indifference curves look like in state-claims space? What do the indifference curves look like in state-claims space? Is risk aversion of order one or of order two?

20. Define $u(y)$ as follows:

$$u(y) = \begin{cases} y & \text{for} \quad y \le 100 \\ 50 + \frac{1}{2} y & \text{for } y \ge 100. \end{cases}$$

Consider a two-state world with an equal probability for both states.
Draw indifference curves for $Eu = 90$, $Eu = 100$ and $Eu = 110$. Is risk aversion of order one or of order two? Be sure to explain your *neatly drawn* diagram.

21. Consider the adverse-selection model of Rothschild and Stiglitz and suppose that there are an equal number of good risks and bad risks. Under the pair of separating contracts that determines a Rothschild-Stiglitz separating equilibrium, the good risks would receive 30% coverage. At a fair pooling price, the optimal level of insurance coverage for the good risks would pay for 70% of the loss. Now suppose that the good risks are indifferent between these two contracts: 70% coverage at the fair pooling price and 30% coverage at the fair good-risk price. Consider the following two potential markets:
(i) all insurers offer only the Rothschild-Stiglitz pair of separating contracts.
(ii) all insurers offer only pooling contracts for 70% coverage at a fair price.

(a) Can either (i) or (ii) define the Rothschild-Stiglitz equilibrium for this market? Explain fully.
(b) Can either (i) or (ii) define the Wilson equilibrium for this market? Explain fully.

22. Consider the Rothschild-Stiglitz adverse-selection model, but with 3 types of insureds: good, medium, and bad, where $p_G < p_M < p_B$. Characterize the R-S equilibrium. Be sure to consider pooling contracts, separating contracts and <u>mixed</u> contracts (i.e. where two types pool with a separate contract for the third type.)

23. Consider a two-state model of insurance demand. The insured has two possible levels of effort. Suppose insurance is offered, but at a price that includes a premium loading $\lambda > 0$. The loading λ can be thought of as a competitive loading to cover marketing expenses. Characterize insurance prices and insurance demand under moral hazard.