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Abstract

We define probability premiums for higher-order risks in Eeckhoudt and Schlesinger's (2006) model of risk apportionment, and show sufficient conditions for comparing the probability premiums between two individuals. For small risk, an individual who is more risk-averse and more prudent will have a larger 3rd-order risk premium. To compare 3rd-order risk premiums for large risk, we use a measure called fear of pain, which can be derived from Friedman and Savage's (1948) utility premium function. We also generalize the fear of pain measure in order to compare 4th-, 5th-, and 6th-order probability premiums.

JEL Classification: D81

Keywords: probability premium, risk aversion, prudence, temperance, risk apportionment

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1 Introduction

Arrow (1974) and Pratt (1964) define a local measure of absolute risk aversion and use it to compare the levels of risk aversion across individuals. It is well known that the Arrow-Pratt measure of risk aversion is equivalent to the risk premium of a gamble which is defined as π such that a decision maker is indifferent between the gamble and the expected value of the gamble minus π . Another equivalent concept is the probability premium, which is defined as the additional probability of winning the positive outcome of a zero-mean gamble yielding two outcomes with equal probabilities that makes the decision maker indifferent between the gamble and the status quo. Even though the risk premium concept has been widely used in the literature, it is rare to find a discussion on the probability premium.

In this paper, we use Eeckhoudt and Schlesinger's (2006) framework to define a probability premium for 3rd-order risk (downside risk) and show sufficient conditions for comparing the probability premiums between two individuals. For random variables X and Y , let $[X, Y]$ be a lottery that yields X or Y , each with probability 0.5. Consider two lotteries, $A_3 = [0, -k + \tilde{\varepsilon}]$ and $B_3 = [\tilde{\varepsilon}, -k]$, where k is a positive constant and $\tilde{\varepsilon}$ is a zero-mean risk. Eeckhoudt and Schlesinger show that a prudent decision maker who always prefers B_3 to A_3 has a utility function of which the third derivative is positive. This condition is equivalent to downside risk aversion, defined by Menezes, Geiss, and Tressler (1980). Crainich and Eeckhoudt (2008) introduce a local measure of the intensity of downside risk aversion and show that the measure is equivalent to a risk premium which is defined as the monetary value added to the payoff 0 in A_3 such that an individual is indifferent between A_3 and B_3 . While Crainich and Eeckhoudt look at the risk premiums, we focus on the probability premium concept. We call the additional probability of winning 0 in A_3 that makes an individual indifferent between A_3 and B_3 the prudence probability premium (or the 3rd-order probability premium). A key difference between the Arrow-Pratt probability premium and the prudence probability premium is that the former is defined on preferences over a binary lottery and a risk-free state, while the latter is defined on preferences over two compound lotteries.

For small risk, we show that the Arrow-Pratt measure of risk aversion ($-u''/u'$) and Kimball's (1990) measure of prudence ($-u'''/u''$) of two individuals are sufficient for ranking the prudence probability premiums. Specifically, a decision maker who is more risk-averse and more prudent will have a higher prudence probability premium. Chiu (2005*b*) interprets the measure of prudence and the measure of risk aversion as the strength of downside risk aversion against risk aversion and the strength of risk aversion against preference for first-order stochastic improvement, respectively. We show that an increase in the probability of winning 0 in A_3 can be decomposed into two components so that we can adopt Chiu's interpretations. The above conditions, however, are not sufficient when $\tilde{\varepsilon}$ is large. To compare the prudence probability premiums given any $\tilde{\varepsilon}$, we define a measure called *fear of pain* which can be derived from the utility premium function introduced by Friedman and Savage (1948) and generalized by Eeckhoudt and Schlesinger (2006). The utility premium for the risk $\tilde{\varepsilon}$ at wealth level x is defined as $w_1(x) \equiv Eu(x + \tilde{\varepsilon}) - u(x)$, which is always negative if preferences are risk-averse. Its negative value can be interpreted as the pain associated with taking the risk $\tilde{\varepsilon}$. Eeckhoudt and Schlesinger (2008) use the reaction of the utility premium to changes in wealth to explain the existence of a precautionary motive for saving. In this paper, we use the utility premium to derive the fear of pain from $\tilde{\varepsilon}$, which is defined as $-w_1''/w_1'$, and show that a decision maker who is more risk-averse and has more fear of pain will always have a higher prudence probability premium.

We also show that the utility premium function can be used to compare the probability premiums for 4th-order risk. Given $A_4 = [0, \tilde{\varepsilon} + \tilde{\varepsilon}]$ and $B_4 = [\tilde{\varepsilon}, \tilde{\varepsilon}]$, Eeckhoudt and Schlesinger show that a temperate decision maker who always prefers B_4 to A_4 has a utility function of which the fourth derivative is negative. We define the temperance probability premium (or the 4th-order probability premium) as the additional probability of winning 0 in A_4 such that an individual is indifferent between A_4 and B_4 . We define the measure of pain risk aversion by $-w_1''/w_1'$ and show that a decision maker who is more averse to pain risk and has more fear of pain will always have a higher temperance probability premium. Following Eeckhoudt and Schlesinger's procedure in constructing lotteries that lead to risk apportionment of orders 5 and 6, we define the second-order utility premium function as $w_2(x) \equiv Ew_1(x + \tilde{\varepsilon}) - w_1(x)$

and use it to compare the 5th- and the 6th-order probability premiums in a similar manner. The only difference between our model and Eeckhoudt and Schlesinger's is that we use the same zero-mean risk $\tilde{\varepsilon}$ for every order of risk.

In the literature, Menezes, Geiss, and Tressler (1980) relate downside risk aversion to the third derivative of the utility function. Keenan and Snow (2002 and forthcoming) characterize the strength of downside risk aversion by proposing a measure that combines u'''/u' and the Arrow-Pratt measure of risk aversion, while Modica and Scarsini (2005) do so by strengthening u'''/u' the same way Ross (1981) does with the Arrow-Pratt measure of risk aversion. Even though Kimball's measure of prudence was initially used to measure the strength of the precautionary saving motive, it has been adopted in many downside risk problems. For example, Chiu (2005*a*) uses the measure of prudence to compare the optimal choice of self-protection expenditure against downside risk. Jindapon and Neilson (2007) show that Modica and Scarsini's condition and Kimball's measure are equivalent to optimal shifts in the distribution function that reduce downside risk under different assumptions. Recently, Crainich and Eeckhoudt (2008) use u'''/u' as a measure of downside risk aversion which is equivalent to the risk premium in risk apportionment of order 3.

Throughout the paper we assume that a decision maker u has a von Neumann-Morgenstern utility function $u(\cdot)$ which is 6 times differentiable and an initial wealth x . In addition to u' , u'' , and u''' , we use $u^{(n)}$ for the n th derivative of the utility function. We assume that $(-1)^n u^{(n)}(x) < 0$ for all x and for $n = 1, 2, \dots, 6$. These conditions are called n th degree risk aversion by Ekern (1980), which is equivalent to risk apportionment of order n defined by Eeckhoudt and Schlesinger (2006). A constant k is strictly positive; a lottery $\tilde{\varepsilon}$ pays ε^+ with probability ρ and ε^- with probability $1 - \rho$, with $\varepsilon^+ > 0$, $\varepsilon^- < 0$ and $E\tilde{\varepsilon} = 0$. We discuss the prudence probability premium given small risk and large risk in Sections 2 and 3, respectively. We extend our main result to temperance and risk apportionment of orders 5 and 6 in Section 3, and summarize in Section 4.

2 Prudence probability premium

Let $A_3 = [0, -k + \tilde{\varepsilon}]$ and $B_3 = [\tilde{\varepsilon}, -k]$ and assume that $\tilde{\varepsilon}$ is small. Eeckhoudt and Schlesinger (2006) say that an individual is prudent if $B_3 \succsim A_3$ for all $x, k, \tilde{\varepsilon}$. We define the prudence probability premium p_3 as additional probability of winning 0 in A_3 such that $A_3 \sim B_3$. Then,

$$(0.5 + p_3)u(x) + (0.5 - p_3)Eu(x - k + \tilde{\varepsilon}) = 0.5Eu(x + \tilde{\varepsilon}) + 0.5u(x - k). \quad (1)$$

We can write

$$2p_3 = \frac{[Eu(x + \tilde{\varepsilon}) - u(x)] - [Eu(x - k + \tilde{\varepsilon}) - u(x - k)]}{u(x) - Eu(x - k + \tilde{\varepsilon})}. \quad (2)$$

Define the utility premium of taking risk $\tilde{\varepsilon}$ as $w_1(x) \equiv Eu(x + \tilde{\varepsilon}) - u(x)$. Then, we have

$$2p_3 = \frac{\Delta w_1(x - k)}{\Delta u(x - k) - w_1(x - k)}, \quad (3)$$

where $\Delta w_1(x - k) = w_1(x) - w_1(x - k)$ and $\Delta u(x - k) = u(x) - u(x - k)$. Eeckhoudt and Schlesinger (2006) prove that $w_1 \leq 0$ if and only if $u'' \leq 0$, and $w_1' \geq 0$ if and only if $u''' \geq 0$. Therefore, if $u'' \leq 0$ and $u''' \geq 0$, then $p_3 \geq 0$.

Dividing the numerator and the denominator in the right-hand side of (3) by $u'(x - k)$ yields

$$2p_3 = \frac{\Delta w_1(x - k)/u'(x - k)}{[\Delta u(x - k) - w_1(x - k)]/u'(x - k)}. \quad (4)$$

Consider another decision maker v with a utility function $v(\cdot)$. Define v 's utility premium function as $z_1(x) \equiv Ev(x + \tilde{\varepsilon}) - v(x)$. Let p_3 and q_3 be the prudence probability premiums for agents u and v , respectively. Hence, for a given $\tilde{\varepsilon}$ and k , u 's prudence probability premium is at least as great as v 's if

$$\frac{\Delta w_1(x - k)}{u'(x - k)} \geq \frac{\Delta z_1(x - k)}{v'(x - k)}, \quad (5)$$

and

$$\frac{\Delta u(x - k) - w_1(x - k)}{u'(x - k)} \leq \frac{\Delta v(x - k) - z_1(x - k)}{v'(x - k)}, \quad (6)$$

for all x . Our first result provides a sufficient condition for (5) and (6).

Proposition 1 Given a small $\tilde{\varepsilon}$, if u is more risk-averse than v , i.e.,

$$-\frac{u''(x)}{u'(x)} \geq -\frac{v''(x)}{v'(x)} \quad (7)$$

for all x , and u is more prudent than v , i.e.,

$$-\frac{u'''(x)}{u''(x)} \geq -\frac{v'''(x)}{v''(x)} \quad (8)$$

for all x , then $p_3 \geq q_3$.

Proof: Part 1. We can write

$$w_1(x) = Eu(x + \tilde{\varepsilon}) - u(x) = \rho \int_x^{x+\tilde{\varepsilon}^+} u'(y)dy - (1 - \rho) \int_{x+\tilde{\varepsilon}^-}^x u'(y)dy, \quad (9)$$

and hence,

$$\begin{aligned} \Delta u(x - k) - w_1(x - k) &= \int_{x-k}^x u'(y)dy - \rho \int_{x-k}^{x-k+\tilde{\varepsilon}^+} u'(y)dy + (1 - \rho) \int_{x-k+\tilde{\varepsilon}^-}^{x-k} u'(y)dy \\ &= \rho \left[\int_{x-k}^x u'(y)dy - \int_{x-k}^{x-k+\tilde{\varepsilon}^+} u'(y)dy \right] + (1 - \rho) \left[\int_{x-k}^x u'(y)dy + \int_{x-k+\tilde{\varepsilon}^-}^{x-k} u'(y)dy \right] \\ &= \rho \int_{x-k+\tilde{\varepsilon}^+}^x u'(y)dy + (1 - \rho) \int_{x-k+\tilde{\varepsilon}^-}^x u'(y)dy = \int_{x-k+\tilde{\varepsilon}^+}^x u'(y)dy + (1 - \rho) \int_{x-k+\tilde{\varepsilon}^-}^{x-k+\tilde{\varepsilon}^+} u'(y)dy. \end{aligned} \quad (10)$$

Dividing (10) by $u'(x - k)$ yields

$$\frac{\Delta u(x - k) - w_1(x - k)}{u'(x - k)} = \int_{x-k+\tilde{\varepsilon}^+}^x \frac{u'(y)}{u'(x - k)}dy + (1 - \rho) \int_{x-k+\tilde{\varepsilon}^-}^{x-k+\tilde{\varepsilon}^+} \frac{u'(y)}{u'(x - k)}dy. \quad (11)$$

Pratt (1964) shows that (7) implies $\frac{u'(y)}{u'(x)} - \frac{v'(y)}{v'(x)} \leq 0$ for all $y > x$, and $\frac{u'(y)}{u'(x)} - \frac{v'(y)}{v'(x)} \geq 0$ for all $y < x$. Therefore,

$$\int_{x-k+\tilde{\varepsilon}^+}^x \frac{u'(y)}{u'(x - k)}dy - \int_{x-k+\tilde{\varepsilon}^+}^x \frac{v'(y)}{v'(x - k)}dy \leq 0. \quad (12)$$

However, the sign of

$$\int_{x-k+\tilde{\varepsilon}^-}^{x-k+\tilde{\varepsilon}^+} \frac{u'(y)}{u'(x - k)}dy - \int_{x-k+\tilde{\varepsilon}^-}^{x-k+\tilde{\varepsilon}^+} \frac{v'(y)}{v'(x - k)}dy \quad (13)$$

is ambiguous. If $\tilde{\varepsilon}$ is small, then (12) dominates (13), and hence (6) is obtained.

Part 2. Consider (9). We find that $\Delta w_1(x - k) =$

$$\int_{x-k}^x w_1'(y) dy = \rho \int_{x-k}^x \int_y^{y+\varepsilon^+} u''(z) dz dy - (1 - \rho) \int_{x-k}^x \int_{y+\varepsilon^-}^y u''(z) dz dy. \quad (14)$$

Dividing (14) by $u''(x - k)$ yields

$$\frac{\Delta w_1(x - k)}{u''(x - k)} = \rho \int_{x-k}^x \int_y^{y+\varepsilon^+} \frac{u''(z)}{u''(x - k)} dz dy - (1 - \rho) \int_{x-k}^x \int_{y+\varepsilon^-}^y \frac{u''(z)}{u''(x - k)} dz dy. \quad (15)$$

Using Pratt's analogy, we find that (8) implies $\frac{u''(y)}{u''(x)} - \frac{v''(y)}{v''(x)} \leq 0$ for all $y > x$, and $\frac{u''(y)}{u''(x)} - \frac{v''(y)}{v''(x)} \geq 0$ for all $y < x$. Therefore,

$$\int_{x-k}^x \int_y^{y+\varepsilon^+} \frac{u''(z)}{u''(x - k)} dz dy - \int_{x-k}^x \int_y^{y+\varepsilon^+} \frac{v''(z)}{v''(x - k)} dz dy \leq 0. \quad (16)$$

However, the sign of

$$\int_{x-k}^x \int_{y+\varepsilon^-}^y \frac{u''(y)}{u''(x - k)} dz dy - \int_{x-k}^x \int_{y+\varepsilon^-}^y \frac{v''(y)}{v''(x - k)} dz dy \quad (17)$$

is ambiguous. If $\tilde{\varepsilon}$ is small, then (16) dominates (17), and hence $\frac{\Delta w_1(x-k)}{u''(x-k)} - \frac{\Delta w_1(x-k)}{v''(x-k)} \leq 0$, which is equivalent to

$$-\frac{\Delta w_1(x - k)}{u''(x - k)} \geq -\frac{\Delta w_1(x - k)}{v''(x - k)}. \quad (18)$$

Since (7) also holds for $x - k$, then (18) implies (5). \square

Proposition 1 states that, when $\tilde{\varepsilon}$ is small, if u is more risk-averse and more prudent than v , then u will have a higher prudence probability premium.¹ This condition is more restrictive than being more downside risk-averse as defined by Crainich and Eeckhoudt (2008). Crainich and Eeckhoudt require $\frac{u'''(x)}{u''(x)} \geq \frac{v'''(x)}{v''(x)}$ for all x which is implied by (7) and (8). Crainich and Eeckhoudt's condition is equivalent to a monetary value m added to the zero outcome of lottery A_3 , while we link our condition to the additional probability of winning zero in A_3 .

Chiu (2005*b*) interprets the measure of prudence and the measure of risk aversion as the strength of downside risk aversion against risk aversion and the strength of risk aversion

¹Eeckhoudt and Schlesinger (1994) discuss the relationship between being more risk-averse and being more prudent and show that it is possible for a more prudent utility function to be more risk-averse or less risk-averse.

against preference for first-order stochastic improvement. Let D_3 represent the lottery that is obtained by adding p_3 to the probability of getting the zero outcome of A_3 . Proposition 1 implies that if $D_3 \sim_u B_3$, and u is more prudent and more risk-averse than v , then $D_3 \succsim_v B_3$. This result can be illustrated by a decomposition of $D_3(y) - B_3(y)$, where $B_3(y)$ and $D_3(y)$ are the distribution functions corresponding to lotteries B_3 and D_3 , respectively. Let $b_3(y)$ and $d_3(y)$ denote the probability density functions of B_3 and D_3 , respectively. Then we have

$$b_3(y) = \begin{cases} 0.5 & \text{if } y = x - k \\ 0.5(1 - \rho) & \text{if } y = x + \varepsilon^- \\ 0.5\rho & \text{if } y = x + \varepsilon^+ \\ 0 & \text{otherwise,} \end{cases}$$

and

$$d_3(y) = \begin{cases} (1 - \rho)(0.5 - p_3) & \text{if } y = x - k + \varepsilon^- \\ \rho(0.5 - p_3) & \text{if } y = x - k + \varepsilon^+ \\ 0.5 + p_3 & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

Let F be a lottery with the density function

$$f(y) = \begin{cases} (1 - \rho)(0.5 - p_3 - \theta) & \text{if } y = x - k + \varepsilon^- \\ \theta & \text{if } y = x - k \\ \rho(0.5 - p_3 - \theta) & \text{if } y = x - k + \varepsilon^+ \\ 0.5(1 - \rho) & \text{if } y = x + \varepsilon^- \\ p_3 & \text{if } y = x \\ 0.5\rho & \text{if } y = x + \varepsilon^+ \\ 0 & \text{otherwise,} \end{cases}$$

where θ is a real number in $(0, 0.5 - p_3)$ such that $F \sim_u D_3$. Since $D_3 \sim_u B_3$, then by transitivity $F \sim_u B_3$. Note that F consists of a mean-preserving contraction around $x - k$ and a mean-preserving spread around x from D_3 . When compare F to B_3 , we find that F consists of a mean-preserving spread around $x - k$ and a first-order stochastic improvement from B_3 . We state Chiu's (2005b) definitions as follows.

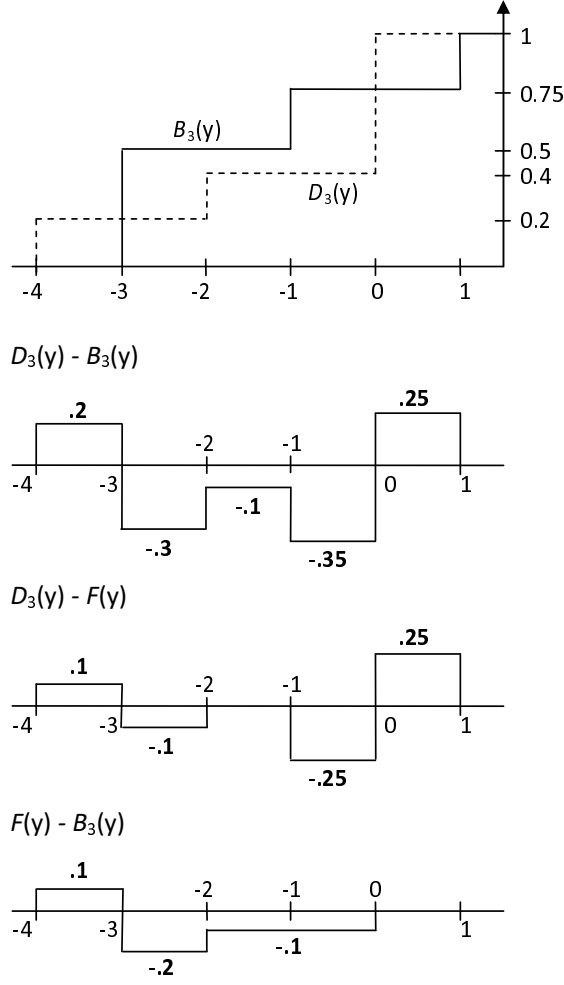


Figure 1: A decomposition of $D_3(y) - B_3(y)$

Definition 1 $F(y) - G(y)$ satisfies Condition 1 if $F(y) - G(y) = -C_1(y) + C_2(y) + C_3(y)$, where $C_1(y)$, $C_2(y)$, and $C_3(y)$ are mean-preserving contractions such that $C_1(y)$ and $C_2(y)$ are of equal size, and $C_1(y)$ precedes $[C_2(y) + C_3(y)]$.

Definition 2 $F(y) - G(y)$ satisfies Condition 2 if $F(y) - G(y) = -I_1(y) + I_2(y) + I_3(y)$, where $I_1(y)$, $I_2(y)$, and $I_3(y)$ are first-order stochastic improvements such that $I_1(y)$ and $I_2(y)$ are of equal size, and $I_1(y)$ precedes $[I_2(y) + I_3(y)]$.

After separating $D_3(y) - B_3(y)$ into two components, $D_3(y) - F(y)$ and $F(y) - B_3(y)$, we find that $D_3(y) - F(y)$ satisfies Condition 1 and $F(y) - B_3(y)$ satisfies Condition 2.

Chiu (2005*b*) proves that (a) if $D_3(y) - F(y)$ satisfies Condition 1 and u is more prudent than v , then $D_3 \sim_u F$ implies $D_3 \succsim_v F$, and (b) if $F(y) - B_3(y)$ satisfies Condition 2 and u is more risk-averse than v , then $F \sim_u B_3$ implies $F \succsim_v B_3$. Thus, by transitivity, $D_3 \succsim_v B_3$. Consider the following example. Let $A_3 = [0, -3 + \tilde{\varepsilon}]$ and $B_3 = [\tilde{\varepsilon}, -3]$ where $\tilde{\varepsilon} = [-1, 1]$. Assume that agent u 's prudence probability premium is equal to 0.1. Then the distributions of B_3 and D_3 are represented by the solid line and the dashed line in the top graph of Figure 1, respectively. The other graphs show that $D_3(y) - B_3(y)$ can be separated into $D_3(y) - F(y)$ and $F(y) - B_3(y)$, where $D_3 \sim_u F \sim_u B_3$. Note that in this example, we assume that $\theta = 0.2$. We find that $D_3(y) - F(y)$ and $F(y) - B_3(y)$ satisfy Conditions 1 and 2, respectively. If u is more prudent and more risk-averse than v , then $D_3 \succsim_v B_3$, and therefore $p_3 \geq q_3$.

3 Large risk

The two conditions in Proposition 1, however, are not sufficient to conclude that $p_3 \geq q_3$ when $\tilde{\varepsilon}$ is large. To compare the prudence probability premiums given any $\tilde{\varepsilon}$, we define a measure called fear of pain which can be derived from the utility premium function introduced by Friedman and Savage (1948) and generalized by Eeckhoudt and Schlesinger (2006). In fact, the fear of pain measure can be easily generalized to compare probability premiums for higher-order risks.

3.1 Prudence probability premium

Dividing the numerator and the denominator in the right-hand side of (3) by $-w_1(x - k)$ yields

$$2p_3 = \frac{-\Delta w_1(x - k)/w_1(x - k)}{1 - \Delta u(x - k)/w_1(x - k)}. \quad (19)$$

We find that u 's prudence probability premium is at least as great as v 's if

$$-\frac{\Delta w_1(x - k)}{w_1(x - k)} \geq -\frac{\Delta z_1(x - k)}{z_1(x - k)}, \quad (20)$$

and

$$-\frac{\Delta u(x-k)}{w_1(x-k)} \leq -\frac{\Delta v(x-k)}{z_1(x-k)}, \quad (21)$$

for all x .

Definition 3 *Agent u has more fear of pain from $\tilde{\varepsilon}$ than agent v if*

$$-\frac{w'_1(x)}{w_1(x)} \geq -\frac{z'_1(x)}{z_1(x)} \quad (22)$$

for all x .²

If u has more fear of pain from $\tilde{\varepsilon}$ than v , then u 's pain from taking $\tilde{\varepsilon}$ increases at a higher rate than v 's at any wealth level. Therefore, if the two individuals lose the same amount of wealth, the one that has more fear of pain will suffer a larger proportion of pain from $\tilde{\varepsilon}$. We use the fear of pain together with the Arrow-Pratt measure of risk aversion to state the following proposition.

Proposition 2 *If u is more risk-averse and has more fear of pain from $\tilde{\varepsilon}$ than v , then $p_3 \geq q_3$.*

Proof: From (9), we can write

$$w_1(x-k) = Eu(x-k+\tilde{\varepsilon}) - u(x-k) = \rho \int_{x-k}^{x-k+\varepsilon^+} u'(y)dy - (1-\rho) \int_{x-k+\varepsilon^-}^{x-k} u'(y)dy. \quad (23)$$

Pratt (1964) shows that (7) implies $\frac{u'(y)}{u'(x)} \leq \frac{v'(y)}{v'(x)}$ for all $y > x$, and $\frac{u'(y)}{u'(x)} \geq \frac{v'(y)}{v'(x)}$ for all $y < x$.

Therefore,

$$\frac{\int_{x-k}^{x-k+\varepsilon^+} u'(y)dy}{u'(x-k)} \leq \frac{\int_{x-k}^{x-k+\varepsilon^+} v'(y)dy}{v'(x-k)}, \quad (24)$$

and

$$\frac{\int_{x-k+\varepsilon^-}^{x-k} u'(y)dy}{u'(x-k)} \geq \frac{\int_{x-k+\varepsilon^-}^{x-k} v'(y)dy}{v'(x-k)}. \quad (25)$$

²Our definition of fear of pain is reminiscent of fear of ruin defined by Aumann and Kurz (1977); u has more fear of ruin than v if $u(x)/u'(x) \geq v(x)/v'(x)$ for all x . However, the fear pain does not have the same properties as the fear of ruin discussed in Foncel and Treich (2005) because $w_1(0)$ and $z_1(0)$ cannot be normalized to 0.

From (23), (24), and (25), we have

$$-\frac{w_1(x-k)}{u'(x-k)} \geq -\frac{z_1(x-k)}{v'(x-k)}. \quad (26)$$

Pratt (1964) also shows that (7) implies $\frac{u'(x)}{u(y)-u(x)} \geq \frac{v'(x)}{v(y)-v(x)}$, for $y > x$. Then,

$$\frac{u'(x-k)}{\Delta u(x-k)} \geq \frac{v'(x-k)}{\Delta v(x-k)}. \quad (27)$$

Using (26) and (27), we have

$$-\frac{w_1(x-k)}{\Delta u(x-k)} \geq -\frac{z_1(x-k)}{\Delta v(x-k)}, \quad (28)$$

and (21) is obtained. Integrating (22) from $y = x + k$ to x yields

$$-[\ln(-w(x)) - \ln(-w(y))] \geq -[\ln(-z(x)) - \ln(-z(y))], \quad (29)$$

which implies

$$-\frac{w_1(x)}{w_1(y)} \geq -\frac{z_1(x)}{z_1(y)}. \quad (30)$$

Adding 1 to each side of (30) yields (20). \square

In general, we need to know the utility function u and the zero-mean risk $\tilde{\varepsilon}$ to derive the fear of pain measure. However, with a CARA utility function $u(x) = -e^{-ax}$, where $a > 0$, we have $w_1(x) = -e^{-ax}[\rho e^{-a\varepsilon^+} + (1 - \rho)e^{a\varepsilon^-} - 1]$. Therefore, the fear of pain measure is equal to a for any x and $\tilde{\varepsilon}$. Given another utility function $v(x) = -e^{-bx}$, where $a > b > 0$, we can say that u is more risk averse and has more fear of pain than v . Hence u 's prudence probability premium is larger than v 's for any $\tilde{\varepsilon}$.

3.2 Higher-order probability premiums

Ekern (1980) generalizes the downside risk concept by using higher orders of stochastic dominance to link the n th-degree risk aversion to the sign of the n th derivative of the utility function. Eeckhoudt and Schlesinger (2006) relate the sign of the n th derivative of the utility function to the n th-order risk apportionment.³ For the 4th-order risk apportionment, we

³Other higher-order generalizations include Caballé and Pomansky (1996), Jindapon and Neilson (2006), and Eeckhoudt, Schlesinger, and Tsetlin (forthcoming).

consider two compound lotteries: $A_4 = [0, \tilde{\varepsilon} + \tilde{\varepsilon}]$ and $B_4 = [\tilde{\varepsilon}, \tilde{\varepsilon}]$. Eeckhoudt and Schlesinger (2006) say that an individual is temperate if $B_4 \succsim A_4$ for all x and $\tilde{\varepsilon}$. We define the temperance probability premium p_4 as the additional probability of winning 0 in A_4 such that $A_4 \sim B_4$. Then,

$$(0.5 + p_4)u(x) + (0.5 - p_4)Eu(x + \tilde{\varepsilon} + \tilde{\varepsilon}) = 0.5Eu(x + \tilde{\varepsilon}) + 0.5Eu(x + \tilde{\varepsilon}). \quad (31)$$

We can write

$$2p_4 = \frac{Ew_1(x + \tilde{\varepsilon}) - w_1(x)}{w_1(x) + Ew_1(x + \tilde{\varepsilon})}. \quad (32)$$

Define the second-order utility premium $w_2(x) \equiv Ew_1(x + \tilde{\varepsilon}) - w_1(x)$. Then, we have

$$2p_4 = \frac{w_2(x)}{w_2(x) + 2w_1(x)} = \frac{1}{1 + 2w_1(x)/w_2(x)}. \quad (33)$$

Eeckhoudt and Schlesinger (2006) show that $w_2 \leq 0, \forall x$, if and only if $u^{(4)} \leq 0$. Therefore if $u'' \leq 0$ and $u^{(4)} \leq 0$, then $p_4 \geq 0$. Define $z_2(x) \equiv Ez_1(x + \tilde{\varepsilon}) - z_1(x)$. Let p_4 and q_4 be the temperance probability premiums for agents u and v , respectively. Therefore $p_4 \geq q_4$ if

$$\frac{1}{1 + 2w_1(x)/w_2(x)} \geq \frac{1}{1 + 2z_1(x)/z_2(x)}, \quad (34)$$

for all x .

Let $A_5 = [B_3, A_3 + \tilde{\varepsilon}]$, $B_5 = [A_3, B_3 + \tilde{\varepsilon}]$, $A_6 = [B_4, A_4 + \tilde{\varepsilon}]$, and $B_6 = [A_4, B_4 + \tilde{\varepsilon}]$. Eeckhoudt and Schlesinger (2006) say that an individual's preferences satisfy risk apportionment of orders 5 and 6 if $B_5 \succsim A_5$ and $B_6 \succsim A_6$ for all x and $\tilde{\varepsilon}$, respectively. We define the 5th-order probability premium p_5 as the additional probability of winning B_3 in A_5 such that $A_5 \sim B_5$. Then,

$$2p_5 = \frac{[Eu(x + A_3) - Eu(x + A_3 + \tilde{\varepsilon})] - [Eu(x + B_3) - Eu(x + B_3 + \tilde{\varepsilon})]}{Eu(x + B_3) - Eu(x + A_3 + \tilde{\varepsilon})}, \quad (35)$$

and hence

$$2p_5 = \frac{-\Delta w_2(x - k)}{w_2(x - k) + 2w_1(x - k)} = \frac{-\Delta w_2(x - k)/w_2(x - k)}{1 - 2w_1(x - k)/w_2(x - k)}, \quad (36)$$

where $\Delta w_2(x - k) = w_2(x) - w_2(x - k)$. Eeckhoudt and Schlesinger (2006) show that $w_2' \geq 0$ if and only if $u^{(5)} \geq 0$. Therefore, if $u'' \leq 0, u^{(4)} \leq 0$ and $u^{(5)} \geq 0$, then $p_5 \geq 0$. Let p_5 and q_5 be the 5th-order probability premiums for agents u and v , respectively. Therefore $p_5 \geq q_5$ if

$$\frac{-\Delta w_2(x)/w_2(x)}{1 + 2w_1(x)/w_2(x)} \geq \frac{-\Delta z_2(x)/z_2(x)}{1 + 2z_1(x)/z_2(x)}, \quad (37)$$

for all x .

We also define the 6th-order probability premium p_6 as the additional probability of winning B_4 in A_6 such that $A_6 \sim B_6$. Then,

$$2p_6 = \frac{[Eu(x + A_4) - Eu(x + A_4 + \tilde{\varepsilon})] - [Eu(x + B_4) - Eu(x + B_4 + \tilde{\varepsilon})]}{Eu(x + B_4) - Eu(x + A_4 + \tilde{\varepsilon})}, \quad (38)$$

and hence

$$2p_6 = \frac{w_3(x)}{w_3(x) + 3w_2(x) + 2w_1(x)} = \frac{1}{1 + 3w_2(x)/w_3(x) + 2w_1(x)/w_3(x)}, \quad (39)$$

where $w_3(x) \equiv Eu_2(x + \tilde{\varepsilon}) - w_2(x)$. Eeckhoudt and Schlesinger (2006) show that $w_3 \leq 0$ if and only if $u^{(6)} \leq 0$. Therefore, if u'' , $u^{(4)}$, and $u^{(6)} \leq 0$, then $p_6 \geq 0$. Define $z_3(x) \equiv Ez_2(x + \tilde{\varepsilon}) - z_2(x)$. Let p_6 and q_6 be the 6th-order probability premiums for agents u and v , respectively. Therefore $p_6 \geq q_6$ if

$$\frac{1}{1 + 3w_2(x)/w_3(x) + 2w_1(x)/w_3(x)} \geq \frac{1}{1 + 3z_2(x)/z_3(x) + 2z_1(x)/z_3(x)} \quad (40)$$

for all x .

Proposition 3 *Let w_1 and w_2 be the utility premium function and the second-order utility premium function of u , and z_1 and z_2 be the utility premium function and the second-order utility premium function of v , respectively. Consider the following conditions:*

- (i). $-\frac{w_1'(x)}{w_1(x)} \geq -\frac{z_1'(x)}{z_1(x)}$ for all x .
- (ii). $-\frac{w_1''(x)}{w_1'(x)} \geq -\frac{z_1''(x)}{z_1'(x)}$ for all x .
- (iii). $-\frac{w_2'(x)}{w_2(x)} \geq -\frac{z_2'(x)}{z_2(x)}$ for all x .
- (iv). $-\frac{w_2''(x)}{w_2'(x)} \geq -\frac{z_2''(x)}{z_2'(x)}$ for all x .

If (i)-(ii) hold, then $p_4 \geq q_4$; if (i)-(iii) hold, then $p_5 \geq q_5$; if (i)-(iv) hold, then $p_6 \geq q_6$.

Proof: In the proof of Proposition 2, we show that (7) implies $-\frac{w_1(x)}{w_1'(x)} \geq -\frac{z_1(x)}{z_1'(x)}$, for all x (see (26)). Analogously, (ii) implies

$$-\frac{w_2(x)}{w_1'(x)} \geq -\frac{z_2(x)}{z_1'(x)}, \quad (41)$$

for all x . If (i) and (41) hold, then

$$\frac{w_2(x)}{w_1(x)} \geq \frac{z_2(x)}{z_1(x)}, \quad (42)$$

for all x . Therefore, (34) is obtained.

In the proof of Proposition 2, we also show that (i) implies $-\frac{\Delta w_1(x)}{w_1(x)} \geq -\frac{\Delta z_1(x)}{z_1(x)}$, for all x (see (30)). Analogously, (iii) implies

$$-\frac{\Delta w_2(x)}{w_2(x)} \geq -\frac{\Delta z_2(x)}{z_2(x)}, \quad (43)$$

for all x . If (42) and (43) hold, then (37) is obtained.

Since (iv) implies

$$-\frac{w_3(x)}{w_2'(x)} \geq -\frac{z_3(x)}{z_2'(x)}, \quad (44)$$

for all x , then (iii) and (44) imply

$$\frac{w_3(x)}{w_2(x)} \geq \frac{z_3(x)}{z_2(x)}. \quad (45)$$

Also, (42) and (45) imply

$$\frac{w_3(x)}{w_1(x)} \geq \frac{z_3(x)}{z_1(x)}. \quad (46)$$

Therefore, (40) is obtained. \square

From the previous section, when condition (i) holds, we say that u has more fear of pain than v . Since the measure $-w_1''/w_1'$ in condition (ii) has the same functional form as the Arrow-Pratt measure of risk aversion, we say that u is more averse to pain risk than v when (ii) holds. The measure of pain risk aversion is introduced in an early draft of Eeckhoudt and Schlesinger (2006) to characterize risk vulnerability, defined by Gollier and Pratt (1996).⁴ When conditions (iii) and (iv) hold, we say that u has more fear of 2nd-order pain than v and u is more averse to 2nd-order pain risk than v , respectively.

For example, given a CARA utility function $u(x) = -e^{-ax}$, we have the measure of pain risk aversion equal to a for all $\tilde{\varepsilon}$. Since $w_2(x) = w_1(x)[\rho e^{-a\varepsilon^+} + (1 - \rho)e^{a\varepsilon^-} - 1]$, then

⁴Agent u is risk-vulnerable if any unfair background risk makes him behave in a more risk-averse way. Eeckhoudt and Schlesinger (2003) prove that u is risk-vulnerable if and only if $-w_1''(x)/w_1'(x) \geq -u''(x)/u'(x)$ for all x .

$-w'_2(x)/w_2(x) = -w''_2(x)/w'_2(x) = a$ for all $\tilde{\varepsilon}$. Given another utility function $v(x) = -e^{-bx}$, where $a > b > 0$, we can say that u is more averse to pain risk and has more fear of pain than v . Moreover, u is more averse to 2nd-order pain risk and has more fear of 2nd-order pain than v . Hence $p_4 \geq q_4$, $p_5 \geq q_5$, and $p_6 \geq q_6$ for all $\tilde{\varepsilon}$.

4 Summary

We use Eeckhoudt and Schlesinger's (2006) framework to define a probability premium for 3rd-order risk and show sufficient conditions for comparing the probability premiums between two individuals. Given $A_3 = [0, -k + \tilde{\varepsilon}]$ and $B_3 = [\tilde{\varepsilon}, -k]$, where k is a positive constant and $\tilde{\varepsilon}$ is a zero-mean risk, we define the prudence probability premium as the additional probability of winning 0 in A_3 that makes an individual indifferent between A_3 and B_3 . We show that, for a small $\tilde{\varepsilon}$, if u is more risk-averse and more prudent than v , then u 's prudence probability premium is higher than v 's.

For larger risk, we use a measure called fear of pain which is derived from the utility premium function introduced by Friedman and Savage (1948). We find that if u is more risk-averse and has more fear of pain than v , then u 's prudence probability premium is higher than v 's. We also generalize the fear of pain measure in order to compare higher-order probability premiums. For instance, we define the temperance probability premium as the additional probability of winning 0 in $A_4 = [0, \tilde{\varepsilon} + \tilde{\varepsilon}]$ that makes an individual indifferent between A_4 and $B_4 = [\tilde{\varepsilon}, \tilde{\varepsilon}]$. We find that if u is more averse to pain risk and has more fear of pain than v , then u 's temperance probability premium is higher than v 's. We show similar results for 5th- and 6th-order risk.

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