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A. van den Nouweland[2]
S. Tijs[2]
M. Wooders[3]

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Department of Economics, Finance and Legal Studies
College of Commerce and Business Administration
200 Alston Hall
The University of Alabama
Tuscaloosa, Alabama 35487-0224
U. S. A.
Fax: (205) 348-0590

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Axiomatizations of Lindahl and ratio equilibria in public good economies[1]

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A. van den Nouweland[2], S. Tijs[2] , and M. Wooders[3]

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Abstract

In this paper we consider public goods economies and two equilibrium concepts: the Lindahl equilibrium and the ratio equilibrium. We show that both can be characterized based on consistency properties. Characterizations of the concepts turn out to be surprisingly simple and direct.

[1]This paper was initiated at the Intérnational Workshop on Game Theory, held at the University of Illinois at Chicago, August 1994, organized by T.E.S. Raghavan.

[2]Department of Econometrics and CentER for Economic Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

[3]Department of Economics, University of Toronto, Toronto, Ontario M5S 1A1, Canada.

1 Introduction

Public goods present major problems in economic theory. A (pure) public good is a commodity that can be consumed in its entirety by all the members of an economy. The Lindahl equilibrium, introduced in Lindahl (1919), and discussed in Samuelson (1954), has the property that individuals pay personalized prices for public goods. Foley (1970) expanded this equilibrium concept to encompass lump sum taxes. In Kaneko (1977a,b), lump sum taxes were required to be proportions of income.

The Lindahl equilibrium for economies with public goods was meant to be a counterpart to the Walrasian equilibrium for private goods economies.[4] On first inspection, the two concepts appear to be quite distinct. The Walrasian equilibrium postulates a single price for each commodity while the Lindahl equilibrium postulates a possibly different price for each consumer of a given public good. Moreover, the Lindahl price paid by an individual depends on his tastes. In spite of the fact that there are important distinctions between the two concepts, there has been significant parallel development in the theories of equilibrium in economies with public goods and with private goods. For example, for both equilibrium concepts, questions of existence, First and Second Welfare Theorems and the relationship to the core have been studied. Recently, Walrasian equilibrium has been studied from the viewpoint of consistency. In this paper we address the consistency of both the Lindahl equilibrium and the related ratio equilibrium.

The study of the consistency properties of solution concepts in economics, game theory, and social choice was initiated by Davis and Maschler (1965). Converse consistency was first examined by Peleg (1986). The consistency principle states that methods of reaching agreements should be consistent whatever the group of agents involved. More precisely, whenever the members of a decision-making situation have accepted an agreement using some particular method, no subgroup of agents, given the acceptance of the complementary coalition and using the same method, has an incentive to reach a different agreement. The consistency principle has been applied to a number of game theoretic and economic solution concepts including the core (Peleg (1985,1986), Winter and Wooders (1994) and several others), the Nash equilibrium (Peleg and Tijs (1993)), the Shapley value (Hart and Mas-Colell (1989)), and the Walrasian equilibrium (van den Nouweland et al. (1994)). A more complete discussion of the literature on consistency is provided in Thomson (1990).

The outline of this paper is as follows. We introduce the model of a public good

[4] See, for example, the discussion in Mas-Colell and Silvestre (1989).

economy in section 2. In section 3, we provide the definition of the ratio equilibrium and we study its consistency properties. The Lindahl equilibrium is defined and studied in section 4. The last section contains our conclusions.

2 Public good economies

In this section we provide the formal definitions of a public good economy and of some associated concepts. Throughout the paper, we restrict discussion to public good economies with one public good and one private good. Our results, however, hold for public good economies with any finite number of public goods. We choose not to consider this broader framework in order to avoid unnecessarily complicated notations.

A *public good economy* (with one private good and one public good) is a list $E = \langle N; (w^i)_{i\in N}; (u^i)_{i\in N}; f\rangle$, where N (sometimes denoted $N(E)$) is a non-empty finite set of agents, $w^i \in \mathbf{R}_+$ is the nonnegative initial endowment of agent $i \in N$ of a private good, $u^i : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$ is the utility function of agent $i \in N$, and $f : \mathbf{R}_+ \to \mathbf{R}_+$ is the cost function for the production of a public good. If agent $i \in N$ has an amount x^i of the private good and the amount of public good that is provided is y, then agent i enjoys a utility of $u^i(x^i, y)$. Further, if each agent i contributes an amount z^i of the private good for the production of the public good, then a bundle y of public good can be provided. The bundle y satisfies $f(y) \leq \sum_{i\in N} z^i$. The family of all public good economies is denoted by $\mathcal{E}$.

An *allocation* in a public good economy is a vector $(x^1, x^2, \ldots, x^n; y)$, where x^i is the amount of private good of agent i for each $i \in N$, and y is the amount of public good that is produced using the contributed amounts of private good, i.e. $f(y) = \sum_{i\in N}(w^i - x^i)$. Formally, denote the set of allocations in an economy E by $A(E)$; thus

$$A(E) = \{((x^i)_{i\in N}; y) \mid x^i \in \mathbb{R}_+ \text{ for each } i \in N,\ y \in \mathbb{R}_+, \text{ and } f(y) = \sum_{i\in N}(w^i - x^i)\}$$

for each $E \in \mathcal{E}$. An allocation $(\mathbf{x}; y) = ((x^i)_{i\in N}; y) \in A(E)$ is *efficient* or *Pareto optimal* for E if there is no other allocation $((\overline{x}^i)_{i\in N}; \overline{y}) \in A(E)$ such that $u^i(\overline{x}^i, \overline{y}) \geq u^i(x^i, y)$ for all $i \in N$ with strict inequality for at least one $i \in N$.

3 A characterization of the ratio equilibrium

In this section we will provide an axiomatic characterization of the ratio equilibrium based on consistency and converse consistency. Both of these axioms treat reduced economies.

We start with the definition of the ratio equilibrium. A ratio equilibrium consists of a level of public good production and a set of ratios, one for each agent in the economy. These ratios reflect the way agents share the cost of public good production; if agents $1, 2, \ldots, n$ have ratios $r^1, r^2, \ldots, r^n$, respectively, then agent i pays the share $\frac{r^i}{\sum_j r^j}$ of the cost of public good production. A set of ratios and a level of public good production constitute a ratio equilibrium if for each agent in the economy the level of public good is such that the utility of the agent is maximized under the restriction that he has to pay a share of the cost of public good production that is reflected by his personalized ratio. Formally, for an $\bar{r}^i \in (0,1]$, the demand set of agent i given that he has to pay a share $\bar{r}^i$ of the cost of producing public good is given by

$$D^i(\bar{r}^i) := \{y \in \mathbb{R}_+ \mid \bar{r}^i f(y) \leq w^i \text{ and } y \text{ maximizes } u^i(w^i - \bar{r}^i f(\bar{y}), \bar{y}) \text{ over all } \bar{y} \geq 0 \text{ such that } \bar{r}^i f(\bar{y}) \leq w^i\}$$

For a public good economy $E = \langle N; (w^i)_{i\in N}; (u^i)_{i\in N}; f\rangle$, the set of *ratio equilibria* is given by $R(E) = \{(\mathbf{r}, y) \in \Delta^n \times \mathbb{R}_+ \mid y \in D^i(\frac{r^i}{\sum_{j\in N} r^j}) \text{ for all } i \in N\}$, where $D^i(\frac{r^i}{\sum_{j\in N} r^j})$ is the demand set of agent i with respect to the ratios $r^1, r^2, \ldots, r^n$, and where we use the notation $\Delta^n := \{(q^1, q^2, \ldots, q^n) \mid q^i \in (0,1] \text{ for each } i \in N\}$.

The ratio equilibrium is basically a mapping that assigns to each public good economy $E \in \mathcal{E}$ a subset $\phi(E)$ of $\Delta^n \times \mathbb{R}_+$. For ease of notation we will call such a mapping a *solution* on $\mathcal{E}$. We will characterize the ratio equilibrium as a solution. In order to study consistency and converse consistency of ratio equilibria, we introduce reduced economies.

Let $E = \langle N; (w^i)_{i\in N}; (u^i)_{i\in N}; f\rangle$ be a public good economy and let $S \subseteq N$, $S \neq \emptyset$, and $(\mathbf{r}, y) \in \Delta^n \times \mathbb{R}_+$. The *reduced economy* of E with respect to S and $(\mathbf{r}, y)$ is

$$E^{S,(\mathbf{r},y)} = \langle S; (w^i)_{i\in S}; (u^i)_{i\in S}; h\rangle,$$

where

$$h(\bar{y}) = \frac{\sum_{i\in S} r^i}{\sum_{i\in N} r^i} f(\bar{y})$$

for all $\bar{y} \in \mathbb{R}_+$. The interpretation of the reduced economy is the following: suppose the agents agree on the level y of the public good and on the division of the costs of production of a level of public good according to the ratios r^i. Then, if the agents in $N\backslash S$ withdraw from the decision-making process, the agents in S can reconsider the level of public good that they are going to produce. If they do so, then they will take into account the fact that agents in $N\backslash S$ will pay a share $\frac{\sum_{i\in N\backslash S} r^i}{\sum_{i\in N} r^i}$ of the cost of producing the public good. Hence, when reconsidering the production of public good, the agents in S face the cost function h. Obviously, the reduced economy is itself an element of $\mathcal{E}$.

Now, we can introduce the consistency property. A solution ϕ on $\mathcal{E}$ is *consistent* (CONS) if it satisfies the following condition. If $E \in \mathcal{E}$, $S \subseteq N(E)$, $S \neq \emptyset$, and $(\mathbf{r}, y) \in \phi(E)$, then $E^{S,(\mathbf{r},y)} \in \mathcal{E}$ and $(\mathbf{r}^S, y) \in \phi(E^{S,(\mathbf{r},y)})$. Hence, for a consistent solution it holds that once an agreement is reached, the withdrawal of some agents from the decision-making process and the subsequent reconsideration by the remaining agents does not change the outcome of the decision-making process. It is shown in the following lemma that the ratio equilibrium is a consistent solution.

Lemma 3.1 The ratio equilibrium on the family $\mathcal{E}$ of public good economies is consistent.

Proof. Let $E = \langle N; (w^i)_{i\in N}; (u^i)_{i\in N}; f\rangle \in \mathcal{E}$ be a public good economy, let $S \subseteq N$, $S \neq \emptyset$, and let $(\mathbf{r}, y) \in R(E)$. Let h be the cost function of the reduced economy $E^{S,(\mathbf{r},y)}$, i.e. $h(\overline{y}) = \frac{\sum_{i\in S} r^i}{\sum_{i\in N} r^i} f(\overline{y})$. Now note that $\frac{r^i}{\sum_{j\in N} r^j} f(\overline{y}) = \frac{r^i}{\sum_{j\in S} r^j} \frac{\sum_{j\in S} r^j}{\sum_{j\in N} r^j} f(\overline{y}) = \frac{r^i}{\sum_{j\in S} r^j} h(\overline{y})$ for all $\overline{y} \in \mathbb{R}_+$. Hence, using the fact that $(\mathbf{r}, y) \in R(E)$ it easily follows that $(\mathbf{r}^S, y) \in R(E^{S,(\mathbf{r},y)})$. □

Now, let us define converse consistency. A solution ϕ on $\mathcal{E}$ is *converse consistent* (COCONS) if for every $E \in \mathcal{E}$ with at least two agents ($|N(E)| \geq 2$) and for every $(\mathbf{r}, y) \in \Delta^n \times \mathbb{R}_+$ the following condition is satisfied. If for every $S \subseteq N(E)$, $S \notin \{\emptyset, N(E)\}$, it holds that $E^{S,(\mathbf{r},y)} \in \mathcal{E}$ and $(\mathbf{r}^S, y) \in \phi(E^{S,(\mathbf{r},y)})$, then $(\mathbf{r}, y) \in \phi(E)$.

The ratio equilibrium satisfies converse consistency, as is proven in the following lemma.

Lemma 3.2 The ratio equilibrium on the family $\mathcal{E}$ of public good economies satisfies converse consistency.

Proof. Let $E = \langle N; (w^i)_{i\in N}; (u^i)_{i\in N}; f\rangle \in \mathcal{E}$ with $|N| \geq 2$ and let $(\mathbf{r}, y) \in \Delta^n \times \mathbb{R}_+$ such that for every $S \subseteq N$, $S \notin \{\emptyset, N\}$, it holds that $(\mathbf{r}^S, y) \in R(E^{S,(\mathbf{r},y)})$. Then, in particular, $(r^i, y) \in R(E^{\{i\},(\mathbf{r},y)})$ for each $i \in N$. Let $i \in N$ and let h be the cost function of the reduced economy $E^{\{i\},(\mathbf{r},y)}$, i.e. $h(\bar{y}) = \frac{r^i}{\sum_{j\in N} r^j} f(\bar{y})$ for all $\bar{y} \in \mathbb{R}_+$. This implies that $(r^i, y) \in R(E^{\{i\},(\mathbf{r},y)})$ indicates that $y \in D^i(\frac{r^i}{\sum_{j \in N} r^j})$. Since we can perform this operation for every $i \in N$, the result is that $(\mathbf{r}, y) \in R(E)$. □

In order to characterize the ratio correspondence using consistency and converse consistency, we need a starting point, i.e., we need to say something about the solution at the level of one-person economies. Thus, we introduce the following property. A solution ϕ on $\mathcal{E}$ satisfies *one-person rationality* (OPR) if for every one-agent public good economy $E = \langle \{i\}; w^i; u^i; f \rangle \in \mathcal{E}$ it holds that $\phi(E) = \{(r^i, y) \in \Delta^1 \times \mathbf{R}_+ \mid y$ is such that $f(y) \leq w^i$ and $u^i(w^i - f(y), y) \geq u^i(w^i - f(\overline{y}), \overline{y})$ for all $\overline{y} \in \mathbf{R}_+$ with $f(\overline{y}) \leq w^i\}$. The one-person rationality axiom can be interpreted as dictating that the individual agent maximizes his utility given his endowment of the private good and the cost of producing certain amounts of the public good (which is in this case like a private good to the agent).

The following lemma shows how the three axioms CONS, COCONS, and OPR interact.

Lemma 3.3 Let ϕ and ψ be two solutions on $\mathcal{E}$ that both satisfy OPR. If ϕ is consistent and ψ is converse consistent, then it holds that $\phi(E) \subseteq \psi(E)$ for all $E \in \mathcal{E}$.

Proof. We will prove the lemma by induction on the number of agents.
If $E \in \mathcal{E}$ is a one-agent economy, i.e. $|N(E)| = 1$, then it follows from OPR of ϕ and ψ that $\phi(E) = \psi(E)$.
Now, let $E \in \mathcal{E}$ be an economy with n agents and suppose that it has already been proven that $\phi(E) \subseteq \psi(E)$ for all economies with less than n agents. Let $(\mathbf{q}, y) \in \phi(E)$. Then, by CONS of ϕ, we know that $E^{S,(\mathbf{q},y)} \in \mathcal{E}$ and $(\mathbf{q}^S, y) \in \phi(E^{S,(\mathbf{q},y)})$ for all $S \subseteq N(E)$, $S \notin \{\emptyset, N(E)\}$. Hence, it follows from the induction hypothesis that $(\mathbf{q}^S, y) \in \psi(E^{S,(\mathbf{q},y)})$ for all $S \subseteq N(E)$, $S \notin \{\emptyset, N(E)\}$. So, by COCONS of ψ, we know that $(\mathbf{q}, y) \in \psi(E)$. We conclude that $\phi(E) \subseteq \psi(E)$. □

Using lemma 3.3, the proof of theorem 3.4 follows directly.

Theorem 3.4 The ratio equilibrium is the unique solution on $\mathcal{E}$ that satisfies OPR, CONS, and COCONS.

Proof. In lemmas 3.1 and 3.2 we proved that the ratio equilibrium satisfies CONS and COCONS. To show that the ratio equilibrium satisfies OPR, let $E = \langle \{i\}; w^i; u^i; f \rangle \in \mathcal{E}$ be a one-agent public good economy. Note that in a one-agent economy, the single agent present will have to pay fully for each level of "public good" that he wants to have available. Hence, the ratio equilibrium of economy E is $\{(r^i, y) \in \Delta^1 \times \mathbf{R}_+ \mid y \in D^i(1)\}$,

where $D^i(1) = \{y \in \mathbf{R}_+ \mid f(y) \leq w^i$ and $u^i(w^i - f(y), y) \geq u^i(w^i - f(\overline{y}), \overline{y})$ for all $\overline{y} \geq 0$ such that $f(\overline{y}) \leq w^i\}$. This proves that the ratio equilibrium satisfies OPR.
To prove unicity, assume that ϕ is a solution on $\mathcal{E}$ that also satisfies the three foregoing axioms. Let $E \in \mathcal{E}$ be arbitrary. Then, lemma 3.3 shows that $\phi(E) \subseteq \psi(E)$ by CONS of ϕ and COCONS of ψ and that $\psi(E) \subseteq \phi(E)$ by CONS of ψ and COCONS of ϕ. Hence, $\phi(E) = \psi(E)$. □

4 A characterization of the Lindahl equilibrium

In this section we will consider Lindahl equilibria. We will show that these equilibria can be characterized axiomatically using basically the same axioms of consistency and converse consistency that we used to characterize the ratio equilibria in the previous section. The difference between the ratio equilibria and the Lindahl equilibria emerges from the definitions of reduced economies and from different "starting points" (i.e., the solutions at the level of one-person economies).

In order to be able to consider Lindahl equilibria, we have to extend our definition of economies slightly by adding the profit shares of the agents in an economy. Hence, in the current section we will consider economies of the form $E = \langle N; (w^i)_{i\in N}; (u^i)_{i\in N}; (\Theta^i)_{i\in N}; f\rangle$, where the Θ^i's are the exogenously given (nonnegative) profit shares of the agents, which are feasible in the sense that $\sum_{i\in N}\Theta^i \leq 1$. We will denote the set of all such economies by $\mathcal{F}$.

A Lindahl equilibrium consists of a level of public good production and a set of inividualized prices, one for each agent in the economy. The prices reflect how much each agent is willing to pay for one unit of public good. A set of prices and a level of public good production constitute a Lindahl eqilibrium if (1) the production of public good is feasible, and (2) given his personalized Lindahl price and his share of the profits in public good production, each agent maximizes his utility subject to his budget constraint. Formally, if a level y of public good is produced and the agents $1, \ldots, n$ in the economy have individualized prices $p^1, \ldots, p^n$, respectively, then the profits made with public good production are $\sum_{j\in N} p^j y - f(y)$, and agent i gets his share $\Theta^i(\sum_{j\in N} p^j y - f(y))$. Hence, the demand set of agent i given the prices $\mathbf{p}$ is given by

$$D^i(\mathbf{p}) := \{y \in \mathbf{R}_+ \mid p^i y \leq w^i + \Theta^i(\sum_{j\in N} p^j y - f(y)) \text{ and } y \text{ maximizes}$$
$$u^i(w^i - p^i\overline{y} + \Theta^i(\sum_{j\in N} p^j\overline{y} - f(\overline{y})), \overline{y}) \text{ over all } \overline{y} \in \mathbf{R}_+$$

$$\text{with } p^i\overline{y} \leq w^i + \Theta^i(\sum_{j\in N} p^j\overline{y} - f(\overline{y}))\}$$

For a public good economy $E = \langle N; (w^i)_{i\in N}; (u^i)_{i\in N}; (\Theta^i)_{i\in N}; f\rangle \in \mathcal{F}$, the set of *Lindahl equilibria* is given by $L(E) = \{(\mathbf{p}, y) \in \Delta^n \times \mathbb{R}_+ \mid y \in D^i(\mathbf{p}) \text{ for all } i \in N\}$.

As we will show in the following, the Lindahl equilibrium on $\mathcal{F}$ can be characterized using basically the same axioms of consistency and converse consistency that we previously encountered in the axiomatic characterization of the ratio equilibrium. However, we must change the definition of reduced economies, because now we have to reduce not with respect to ratio equilibria, but with respect to Lindahl equilibria.

Let $E = \langle N; (w^i)_{i\in N}; (u^i)_{i\in N}; (\Theta^i)_{i\in N}; f\rangle \in \mathcal{F}$ be a public good economy and let $S \subseteq N$, $S \neq \emptyset$, and $(\mathbf{p}, y) \in \Delta^n \times \mathbf{R}_+$. The *reduced economy* of E with respect to S and $(\mathbf{p}, y)$ is

$$E^{S,(\mathbf{p},y)} = \langle S; (w^i)_{i\in S}; (u^i)_{i\in S}; (\Theta^i)_{i\in S}; g\rangle,$$

where

$$g(\overline{y}) := f(\overline{y}) - \sum_{i\in N\setminus S} p^i\overline{y}$$

for all $\overline{y} \in \mathbf{R}_+$. The interpretation of the reduced economy $E^{S,(\mathbf{p},y)}$ is the following: suppose the agents in N agreed on the level y of public good and on individualized prices $\mathbf{p}$. Then, if the agents in $N\setminus S$ withdraw from the decision-making process, the agents in S can reconsider the level of public good that they are going to produce. If they do so, then they will take into account that the agents in $N\setminus S$ will pay their respective prices for each unit of public good that is produced and that these agents will get their shares of the profits that are made with public good production. Hence, when reconsidering the production of public good, the agents in S face cost function g. Obviously, the reduced economy is itself an element of $\mathcal{F}$.

The notions of a solution on $\mathcal{F}$ and of consistency and converse consistency for solutions on $\mathcal{F}$ are straightforeward adaptions of the corresponding definitions in the previous section; thus we leave this to the reader.

It is shown in the next lemma that the Lindahl equilibrium satisfies consistency.

Lemma 4.1 The Lindahl equilibrium on the family $\mathcal{F}$ of public good economies is consistent.

Proof. Let $E = \langle N; (w^i)_{i\in N}; (u^i)_{i\in N}; (\Theta^i)_{i\in N}; f\rangle \in \mathcal{F}$ be a public good economy. Let $S \subseteq N$, $S \neq \emptyset$, and let $(\mathbf{p}, y) \in L(E)$. It is easily seen that $E^{S,(\mathbf{p},y)} \in \mathcal{F}$. Let g

be the cost function of the reduced economy $E^{S,(\mathbf{p},y)}$, i.e. $g(\overline{y}) = f(\overline{y}) - \sum_{i \in N \setminus S} p^i \overline{y}$. Then it holds for all $(\mathbf{p}^S, \overline{y}) \in \Delta^s \times \mathbb{R}_+$ that $\sum_{i \in S} p^i \overline{y} - g(\overline{y}) = \sum_{i \in S} p^i \overline{y} - f(\overline{y}) + \sum_{i \in N \setminus S} p^i \overline{y} = \sum_{i \in N} p^i \overline{y} - f(\overline{y})$. Hence, it follows from the fact that $(\mathbf{p}, y) \in L(E)$ that $(\mathbf{p}^S, y) \in L(E^{S,(\mathbf{p},y)})$. □

The following lemma shows that the Lindahl equilibrium also satisfies converse consistency.

Lemma 4.2 The Lindahl correspondence is converse consistent on the family $\mathcal{F}$ of public good economies.

Proof. Let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; (\Theta^i)_{i \in N}; f \rangle \in \mathcal{F}$ with $|N| \geq 2$ and let $(\mathbf{p}, y) \in \Delta^n \times \mathbb{R}_+$ such that for every $S \subseteq N$, $S \notin \{\emptyset, N\}$, it holds that $(\mathbf{p}^S, y) \in L(E^{S,(\mathbf{p},y)})$. Then, in particular, $(p^i, y) \in L(E^{\{i\},(\mathbf{p},y)})$ for each $i \in N$. Let $i \in N$ and let g be the cost function of the reduced economy $E^{\{i\},(\mathbf{p},y)}$, i.e. $g(\overline{y}) = f(\overline{y}) - \sum_{j \in N \setminus \{i\}} p^j \overline{y}$ for all $\overline{y} \in \mathbf{R}_+$. Then, for each $\overline{y} \in \mathbf{R}_+$ it holds that $p^i \overline{y} - g(\overline{y}) = \sum_{j \in N} p^j \overline{y} - f(\overline{y})$. This implies that $(p^i, y) \in L(E^{\{i\},(\mathbf{p},y)})$ indicates that $y \in D^i(\mathbf{p})$. Since we can do this for every $i \in N$, it results that $(\mathbf{p}, y) \in L(E)$. □

In order to characterize the Lindahl equilibrium using consistency and converse consistency, we need one more axiom, namely an axiom that states what happens in one-person economies. A solution ϕ on $\mathcal{F}$ satisfies *one-person maximization* (OPM) if for every one-agent public good economy $E = \langle \{i\}; w^i; u^i; \Theta^i; f \rangle \in \mathcal{F}$ it holds that the individual agent maximizes his utility given his initial endowment of the private good, his personalized Lindahl price for the public good, and his share of the profits from public good production. In formula, this means that $\phi(E) = L(E)$.

The following lemma, which shows how the three axiom CONS, COCONS, and OPM interact, is the analogon of lemma 3.3. The proof of lemma 4.3 is left to the reader.

Lemma 4.3 Let ϕ and ψ be two solutions on $\mathcal{F}$ that both satisfy OPM. If ϕ is consistent and ψ is converse consistent, then it holds that $\phi(E) \subseteq \psi(E)$ for all $E \in \mathcal{F}$.

Using lemmas 4.1, 4.2, and 4.3 we obtain an axiomatic characterization of the Lindahl equilibrium.

Theorem 4.4 The Lindahl equilibrium is the unique solution on $\mathcal{F}$ that satisfies OPM, CONS, and COCONS.

Proof. Obviously, the Lindahl correspondence satisfies OPM. Further, in lemmas 4.1 and 4.2 we proved that the Lindahl correspondence satisfies CONS and COCONS.

Now, assume that ϕ is a solution on $\mathcal{F}$ that also satisfies the three foregoing axioms. Let $E \in \mathcal{F}$ be arbitrary. Then lemma 4.3 shows that $\phi(E) \subseteq \psi(E)$ by CONS of ϕ and COCONS of ψ and that $\psi(E) \subseteq \phi(E)$ by CONS of ψ and COCONS of ϕ. Hence, $\phi(E) = \psi(E)$. □

5 Conclusions

Our work provides parallels between the theories of public and private goods provision. Our axioms are remarkably similar to those for the Walrasian equilibrium, first given by van den Nouweland et al. (1994). Note that the characterizations of the ratio and the Lindahl equilibria have one less axiom than the characterization of the Walrasian equilibrium. The extra axiom in the axiomatization in the latter case treats two-person economies and has the consequence that all individuals face the same prices for the same commodities. In the axiomatizations of the ratio and the Lindahl equilibria we do not need such an axiom, because the ratios and the prices are individualized.

In this paper we have considered a special model in the sense that there is only one private good and one public good. As remarked in the text, extending the model to more than one public good is simple. Adding more private goods does not appear to be enlightening – we arrive at the problem of axiomatizing private goods economies found in van den Nouweland et al. (1994) and additional problems of public goods economies without reaching any new conclusions.

A more interesting research topic may be to consider other equilibrium concepts in other environments from a consistency perspective. Notions of equilibria for public good economies that have recently been introduced include cost share equilibria by Mas-Colell and Silvestre (1989). For economies with public goods subject to crowding, some equilibrium concepts posit prices for public goods based on observable characteristics of participants; see Conley and Wooders (1994). Other equilibrium concepts posit prices for public goods based on preferences, as in this paper, Wooders (1994), Scotchmer and Wooders (1986), Scotchmer (1994), and Vasil'ev, Weber, and Wiesmeth (1992), for example. Conley and Wooders (1994) establish that in large economies where small groups are effective for the realization of all gains to collective activities, equilibrium outcomes with prices for public goods based on crowding characteristics of participants are equivalent to equilibrium outcomes with prices for public goods based on preferences. This equivalence suggests that economies with crowded public goods may be particularly interesting situations in which to compare the axiomatic properties of equilibrium concepts. We plan to treat this topic in future research.

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