LARGE SAMPLE TESTS FOR ABSOLUTE LORENZ DOMINANCE

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We propose statistical tests for comparing the absolute Lorenz curves introduced by Moyes (1987). These tests allow absolute inequality comparisons using the tools of statistical inference. We apply the tests to US state income distributions, obtaining rankings in 96 percent of the comparisons.

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1. Introduction

In a recent paper, Moyes(1987) introduced the "absolute Lorenz curve" as a device for evaluating income inequality. The absolute Lorenz curve is invariant to equal increments to all incomes, and corresponds to the concept of inequality referred to by Kolm (1976a, b) as "leftist," and by Blackorby and Donaldson (1980) as "absolute" inequality. Absolute Lorenz dominance, where the absolute Lorenz curve for income distribution A lies everywhere above the absolute Lorenz curve for distribution B, is a subrelation of all absolute (translation invariant) inequality orderings.

The purpose of this paper is to propose statistical test procedures for comparing absolute Lorenz curves. These tests allow the comparison of different income distributions by the criterion relevant for absolute inequality rankings using the tools of statistical inference.[1] The tests presented here are based on a modification of the asymptotically distribution-free tests for generalized Lorenz dominance (Bishop, Chakraborti, and Thistle, 1987).[2]

We illustrate the tests by ranking the individual states relative to the United States as a whole by the absolute Lorenz criterion. We contrast these results with

1. Without a statistical test, any judgement about the inequality of incomes from sample data, even on the basis of this appealing criterion, is bound to be speculative and hence potentially misleading.
2. Generalized Lorenz curves are introduced in Shorrocks (1983).

the ranking obtained by numerical comparisons. The statistical tests allow a more complete ordering, thus strengthening the underlying economic theory.

2. Definitions and the Testing Problem.

For a distribution F with mean μ, the absolute Lorenz curve is defined as

$$LA(p) := \int_0^y (x - \mu)\,dF(x),\ p \in [0, 1], \qquad 2.1$$

where $p = F(y)$. As Moyes observes, the absolute Lorenz curve can easily be written in terms of either the ordinary Lorenz curve or the generalized Lorenz curve. Then the absolute Lorenz curve is

$$LA(p) = \mu(LR(p) - p)$$

or

$$LA(p) = GL(p) - p\mu, \qquad 2.2$$

where $LR(p)$ is the relative (ordinary) Lorenz ordinate, and $GL(p)$ is the generalized Lorenz ordinate at $p \in [0, 1]$. Recognizing that $GL(1) = \mu$, then $LA(p) = GL(p) - pGL(1)$. Income distribution A absolute Lorenz dominates distribution B iff $LA^A(p) \geq LA^B(p)$ on $[0, 1]$.

Our objective is to derive statistical tests for the equality of two sets of LA ordinates. The LA ordinates are estimated at the K fractions, $0<p_1<p_2<\ldots<p_K<1$, where $p_{K+1} = 1$. Letting LA_i and GL_i be the absolute Lorenz and generalized Lorenz ordinates at p_i, 2.2 can be written as

$$LA_i = GL_i - p_i GL_{K+1}. \qquad 2.3$$

The relationship 2.3 plays a central role in the derivation of the proposed tests.

Let $\mathbf{LA}^A = (LA_1^A, LA_2^A, \ldots, LA_K^A)'$ and $\mathbf{LA}^B = (LA_1^B, LA_2^B, \ldots, LA_K^B)'$ denote the K-vectors of absolute Lorenz ordinates for income distributions A and B.[3] We may want to test the null hypothesis

$$H_0^1 : \mathbf{LA}^A = \mathbf{LA}^B.$$

This will be referred to as an overall hypothesis.[4] One may be also interested in locating at which of the K ordinates the absolute Lorenz curves differ. This can be accomplished by testing the null hypothesis

$$H_o^2 : LA_i^A = LA_i^B, \text{ for } i=1,2,\ldots,K.$$

We assume that two microdata samples of N_A and N_B observations are available, and both the measure of income and the receipient unit are chosen appropriately.

3. The Proposed Tests

Let ξ_i be the p_ith population quantile, $F(\xi_i) = p_i$. The conditional mean and variance of incomes less than ξ_i are $\gamma_i = E(y|\ y \le \xi_i)$ and $\lambda_i^2 = E[(y - \gamma_i)^2|\ y \le \xi_i]$. Observe that γ_{K+1} and λ_{K+1}^2 are the unconditional mean and the variance. Assume that F is twice differentiable and strictly monotonic, and that γ_{K+1} and λ_{K+1}^2 exist.

3. Since LA(1) = 0, only K absolute Lorenz ordinates are freely variable.
4. If the distributions are known to have the same mean, then the null hypothesis is equivalent to equality of the ordinary Lorenz ordinates. In this case the test proposed by Beach and Davidson (1983) may also be applied.

The i^{th} generalized Lorenz ordinate is $GL_i = p_i\gamma_i$, which can be consistently estimated by $\hat{G}L_i = (1/N) \sum^{ri} x_j$, where $r_i = [Np_i]$ is the integer part of Np_i and x_j is the j^{th} order statistic. Then $\hat{\mathbf{GL}}$ is asymptotically normally distributed.

Theorem 1: (Beach and Davidson, 1983). Under the above assumptions, $N^{1/2}(\hat{\mathbf{GL}} - \mathbf{GL})$ has a limiting (K+1)-variate normal distribution with mean zero and dispersion matrix W, where, for $i \le j$,

$$W_{ij} = p_i[\lambda_i^2 + (1-p_j)(\xi_i - g_i)(\xi_j - \gamma_j) + (\xi_i - \gamma_i)(\gamma_j - \gamma_i)] \qquad 3.1$$

The i^{th} absolute Lorenz ordinate can be estimated by $\hat{L}A_i = \hat{G}L_i - p_i\, \hat{G}L_{K+1}$, $i=1,2,\ldots,K$. The limiting distribution of $\hat{\mathbf{LA}}$ follows from Theorem 1.

Theorem 2: Under the above assumptions, $N^{1/2}(\hat{\mathbf{LA}} - \mathbf{LA})$ has a limiting K-variate normal distribution with mean zero and dispersion matrix Q, where, for $i \le j$,

$$Q_{ij} = W_{ij} + p_i p_j W_{K+1,K+1} - p_i W_{jK+1} - p_j W_{iK+1}. \qquad 3.2$$

The W_{ij} can be consistently estimated by estimating the ξ_i, γ_i and λ_i by their sample analogs, and 3.2 used to consistently estimate Q. Therefore, if $\hat{Q}$ is consistent estimator of Q, a reasonable statistic to to test H_0^1 is the quadratic form

$$T_1 = (\hat{\mathbf{LA}}^A - \hat{\mathbf{LA}}^B)' \, [\hat{Q}]^{-1} \, (\hat{\mathbf{LA}}^A - \hat{\mathbf{LA}}^B). \qquad 3.3$$

It is clear that under H_0^1, the limiting distribution of T_1 is a central chi-square with K degrees of freedom.

In order to test H_0^2, we make use of the statistics

$$T_{2i}=(\hat{L}A_i^A-\hat{L}A_i^B)/[\hat{Q}_{ii}^A/N_A + \hat{Q}_{ii}^B/N_B]^{1/2}, \quad 3.4$$

for i=1,2,...,K. Note that testing H_0^2 is in fact a problem of simultaneous inference. This is because the hypothesis H_0^2 can be viewed as an union of K disjoint sub-hypotheses. Our objective is to test each of these sub-hypotheses simultaneously such that the overall probability of rejecting the null hypothesis remains fixed. Following Bishop, Chakraborti and Thistle(1987), we test H_0^2 by testing each of the statistics T_{2i} as a Studentized Maximum Modulus(SMM) variate.[5]

4. Empirical Analysis.

To demonstrate the usefulness of the proposed inferences procedures, we apply them empirically, using a one quarter percent random sample of households from the 1980 Census of Population.[6,7] We estimate the absolute Lorenz curves for the fifty-one states (including the District of Columbia) and U.S. as a whole at vigintiles (K+1 = 20). We compare each state absolute Lorenz curves to the U.S. as a whole.

First, we carry out the usual numerical comparisons. Forty states can be ranked relative to the U.S. (78.4

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5. To implement the test, tables for the percentiles of the SMM distribution by Stoline and Ury(1979) can be used.
6. Here we use equation 3.4 as it allows us to rank the absolute Lorenz curves as dominating (dominated), no different or intersecting. If one is interested only in determining if two absolute Lorenz curves are the same or different then equation 3.3 is appropriate.
7. For a more complete description of the data used, see Bishop, Formby and Thistle(1987).

percent); the remaining eleven states' absolute Lorenz curves intersect the U.S. absolute Lorenz curve. Of the forty states that can be apparently ranked, thirteen dominate and twenty-seven are dominated by the U.S. as a whole. As the probality of making a Type I error with the "numerical test" is one, numerical comparisons almost surely preclude the finding of no difference between two distributions.

We then apply the statistical test (equation 3.4) at one percent significance level. In contrast to the numerical rankings, forty-nine out of the fifty-one states can be ranked (96.0 percent). Twenty-three states dominate and twelve states are dominated by the U.S. as a whole. Importantly, fourteen states are ranked as no different from the U.S. as a whole by the statistical tests. Six of these "no different" state absolute Lorenz curves numerically intersect the U.S. absolute Lorenz curve.

5. Conclusion

This paper provides distribution-free statistical tests for absolute Lorenz dominance. These tests are computationally simple and applicable in a wide variety of situations. Further, the statistical tests, unlike numerical comparisons, allow the ranking of two distributions as equivalent. These statistical procedures are found to provide a more complete empirical partial ordering of income distributions.

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