THE SEMIVARIANCE, OTHER LOWER PARTIAL MOMENTS, AND EXPECTED UTILITY

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One of the approaches to the choice among risky investments is the mean-semivariance model, originally suggested and partially explored by Markowitz in his 1959 book [11] as an alternative to the mean-variance model. A modification or generalization of the mean-semivariance model to a mean-k-th order lower partial moment model has been suggested recently by Fishburn [3] and by Bawa and Lindenberg [2]. Another class of models based directly on the expected utility hypothesis, the stochastic dominance class, has been developed over approximately the same period (see [12], [4], [5] and [17] for the main development). The analytical and empirical relationships between stochastic dominance rankings and mean-semivariance rankings have been partially explored by Porter [14], Bawa [1], Jean [7], and Fishburn [3]. In the current paper the mathematical relationship between the mean-semivariance model and all degrees of stochastic dominance will be developed, and among other results it will be shown that a semivariance ranking is not a necessary condition for a fourth degree or higher stochastic dominance ranking. The mean-semivariance model is therefore not consistent with a great number of generally accepted investor utility functions, and as a result is likely to lead to erroneous investment decisions and incorrect security price derivations. In this paper the mean-k-th order lower partial moment model is also compared mathematically to higher degree stochastic dominance models and similar results are attained--the k-th order lower partial moment ranking is not a necessary condition for k+2nd or higher degree stochastic dominance. Neither form of lower partial moment model is consistent with decreasing risk aversion of investors.

The principal steps in the development of the mean-semivariance model include Markowitz's [11] exploration in 1959, Mao's [10] 1970 justification of the model on the basis of a piecewise quadratic and linear investor utility function and survey of businessmen decision processes, and Hogan and Warren's [6] 1974 derivation of a capital asset pricing model based on the semivariance. In 1974 Porter [14] proved analytically that a mean-semivariance ranking was a necessary condition for a second degree stochastic dominance ranking, and presented empirical evidence of the relationship. In 1975 Bawa [1] and Jean [7] derived mathematical proofs that the mean-semivariance ranking was necessary for third degree stochastic dominance.

Two definitions are used for the semivariance, the first where deviations are measured from the mean of the distribution, and the second where deviations are measured from an arbitrary point in the range of investment returns. The second definition is more commonly used, and that form will be used in this paper.

The probability density function for a continuous distribution of investment returns $\underline{x}$ over a finite range [a, b] is f(x) for a first investment choice and g(x) for a second choice. The semivariances defined for an arbitrary point $\underline{c}$ ($a \leq c \leq b$) are

$$SV_f(c) = \int_a^c (x-c)^2 f(x)\, dx \quad \text{and} \quad SV_g(c) = \int_a^c (x-c)^2 g(x)\, dx$$

To conform to a notation commonly used in stochastic dominance derivations the successive integrals of the probability density function are $F(x)$, $F_1(x)$ and in general $F_k(x)$. As shown by Jean [7] the semi-variance can be integrated by parts twice and the result is

$$SV_f(c) = 2\ F_2(c)$$

The arithmetic mean of the distribution can be derived by integrating by parts once with the result

$$E_f = \int_a^b x\ f(x)\ dx = b - F_1(b)$$

The mean-semivariance (weak) ranking rule for the first investment to be preferred to the second can therefore be expressed as

$$F_1(b) \leq G_1(b)$$

and $$F_2(c) \leq G_2(c)$$

Stochastic dominance models of various degrees have been based on the expected utility hypothesis that investors choose among mutually exclusive investments on the basis of the mathematical expected utility of the investment returns. The degree of stochastic dominance test depends on the knowledge assumed of the signs of the successive derivatives of the investor's utility function. In first degree stochastic dominance the sign of the first derivative is assumed, in second degree stochastic dominance the signs of the first two derivatives are assumed, and in general n-th degree stochastic dominance assumes the signs of the first $\underline{n}$ derivatives of the utility function.

The validity of assuming the successive signs of derivatives of a utility function was made clearer on the publication of Scott and Horvath's 1980 paper [15]. To arrive at the result that successive derivatives must alternate in signs it is only necessary to assume that individuals have strictly consistent preferences (that is, the derivative is either strictly positive, strictly negative, or zero for all values of the argument of the function), that individuals have strictly positive marginal utilities, and that they exhibit strictly decreasing risk aversion (either absolute or relative) behavior. In an alternate derivation Jean [9] showed that the only necessary assumption is that a risk premium function be completely monotonic.

The first degree stochastic dominance (FSD) ranking rule for the first investment to be (weakly) preferred to the second is

$$F(x) \leq G(x) \text{ for all } x \text{ in the range.}$$

A strong ranking is achieved if the inequality holds strictly for some x value. The second degree stochastic dominance (SSD) ranking is achieved if

$$F_1(x) \leq G_1(x) \text{ for all } x.$$

With third degree stochastic dominance (TSD) two inequalities are sufficient for a ranking

$$F_1(b) \leq G_1(b)$$

and $$F_2(x) \leq G_2(x) \text{ for all } x.$$

As shown by Jean [8] n-th degree stochastic dominance (NSD) is achieved if the n-1 conditions

$$F_k(b) \leq G_k(b) \text{ for } k = 1, 2, \ldots, n-2$$

and $$F_{n-1}(x) \leq G_{n-1}(x) \text{ for all } x$$

are met. An empirical study of portfolios using NSD can be found in Tehranian's [16] 1980 paper.

A lower degree stochastic dominance ranking always implies a higher degree stochastic dominance ranking. By statistical definition F(x) is always non-negative and non-decreasing, so if the integral for the first investment is ordered with the corresponding integral for the second investment for all points in the distribution,

$$F_k(x) \leq G_k(x) \text{ for all } x \ (k = 0, 1, 2, \ldots)$$

successive integrals must also be ordered at all points,

$$F_{k+1}(x) \leq G_{k+1}(x) \text{ for all } x.$$

Higher degree stochastic dominance does not imply lower degree stochastic dominance, of course.

The necessity of a mean-semivariance ranking for first, second, and third degree stochastic dominance is easily shown. If FSD exists between the two distributions

$$F(x) \leq G(x) \text{ for all } x$$

and the successive integrals are ordered:

$$F_1(x) \leq G_1(x) \text{ for all } x, \text{ including } b$$

and $$F_2(x) \leq G_2(x) \text{ for all } x, \text{ including } c.$$

FSD cannot exist without a mean-semivariance ranking. If SSD exists then

$$F_1(x) \leq G_1(x) \text{ for all } x, \text{ including } b$$

and $$F_2(x) \leq G_2(x) \text{ for all } x, \text{ including } c.$$

The mean-semivariance ranking is necessary for SSD. With TSD

$$F_1(b) \leq G_1(b)$$

and $$F_2(x) \leq G_2(x) \text{ for all } x, \text{ including } c.$$

The similarities between the TSD tests and the mean-semivariance ranking tests are easily seen. If the target return $\underline{c}$ is varied over the entire range of $\underline{x}$ the semivariance test would amount to a comparison of $F_2(x)$ with $G_2(x)$ for all $\underline{x}$ values. If $\underline{c}$ is considered to be an unknown parameter and then caused to vary all over the range of $\underline{x}$ the mean-semivariance ranking model becomes the third degree stochastic dominance model.

The ranking of arithmetic means is a necessary condition for fourth and higher degrees of stochastic dominance, but a ranking of semivariances is not. The ranking conditions for fourth degree stochastic dominance (4SD), for example, are

$$F_1(b) \leq G_1(b)$$

$$F_2(b) \leq G_2(b)$$

and $$F_3(x) \leq G_3(x) \text{ for all } x.$$

In the two distributions being compared it is possible that the semivariance be ordered in the wrong direction,

$$SV_f > SV_g$$

or $$F_2(c) > G_2(c),$$

but the inequalities for 4SD can still be met. A pair of distributions could have

$$F_2(x) > G_2(x) \text{ for some x values, including c}$$

and $$F_2(x) \leq G_2(x) \text{ for other x values, including b}$$

and have the next integrals completely ordered,

$$F_3(x) \leq G_3(x) \text{ for all x.}$$

A numerical example should prove that the semivariance ordering is not a necessary condition for 4SD. In the discrete example

x =	1	2	3	4	5	6	7	8	9
First distribution probability	0	0.5	0	0	0	0	0.2	0	0.3
Second distribution probability	0.2	0	0.3	0	0.1	0	0	0.1	0.3

the first distribution dominates the second distribution when the 4SD tests are applied. The means of the two distributions are equal (at 5.1), but the semivariance of the first distribution is larger than the semivariance of the second distribution at certain reference values such as c=5. According to the mean-semivariance model the second distribution should be preferred when that reference value is used, an opposite ranking from the 4SD result. Not only is the semivariance ranking not a necessary condition for 4SD, but opposite results can be obtained from the two models when distributions with equal arithmetic means are compared.

Since fourth degree stochastic dominance implies fifth degree and all higher degrees of stochastic dominance the numerical example is sufficient to prove that the semivariance ranking is not a necessary condition for any degree of stochastic dominance higher than the third degree. For distributions with equal arithmetic means the mean-semivariance model and higher degree stochastic dominance models can yield contradictory rankings, casting considerable doubt on the mean-semivariance model's validity since the stochastic dominance model is based on expected utility.

Fourth degree stochastic dominance requires the specification of signs of the first four derivatives of the decision-maker's utility function. The fourth derivative should be non-positive to allow the weak-form ranking rules to be derived, but can be assumed to be strictly negative for all $\underline{x}$, or at least for some $\underline{x}$ values included in the two distiributions to allow a strong-form ranking. That sign can be derived by the Scott-Horvath [15] procedure, based upon the assumption of decreasing risk aversion.

Many commonly accepted utility functions such as the Bernoulli logarithmic function exhibit decreasing risk aversion and therefore have a negative fourth derivative, so the fourth degree stochastic dominance tests would be consistent with those utility functions. Since the mean-semivariance ranking model can give contradictory results to the fourth degree stochastic dominance model the mean-semivariance model can be said to be inconsistent with a large number of generally accepted utility functions.

The mean-semivariance capital asset pricing model analogue derived by Hogan and Warren [6] has been compared to the mean-variance original by Nantell and Price [12] when certain distributions prevail. It would appear that such a mean-semivariance pricing model's validity must rely on restrictive assumptions about the distributions of security returns or about the form of investor utility functions just as the mean-variance model's validity does. The results presented above indicate that the use of the mean-semivariance criterion may lead to selection of a non-optimum portfolio as a basis for pricing, and focusing on the cosemivariance between the portfolio and an individual security could compound the error. The mean-k-th order lower partial moment capital asset pricing model developed by Bawa and Lindenberg [2] is subject to the same type of possible error.

Fishburn's [3] and Bawa and Lindenberg's [2] 1977 papers suggest the possibility of expanding the mean semivariance ranking model to an expected return-risk model where a higher degree lower partial moment is defined as a substitute for the semivariance to measure risk. The k-th order lower partial moment is

$$M_{f_k}(c) = \int_a^c (x-c)^k \, f(x) \, dx \qquad (k = 1, 2, \ldots)$$

The k-th order lower partial moment can be integrated by parts <u>k</u> times to derive

$$M_{f_k}(c) = (-1)^k \, k! \, F_k(c)$$

A ranking of the k-th order lower partial moments is a necessary condition for k+1st degree stochastic dominance,

$M_{fk}(c) \leq M_{gk}(c)$ for even k

or $M_{fk}(c) \geq M_{gk}(c)$ for odd k,

so that the inequality

$F_k(x) \leq G_k(x)$ for all x

can be met.

The mean-k-th order lower partial moment ranking would not, of course, form a complete set of necessary conditions for k+1st degree stochastic dominance. In fourth degree stochastic dominance the second inequality

$F_2(b) \leq G_2(b)$

is not related to the k-th order lower partial moment, and other necessary conditions for stochastic dominance such as the least value condition derived by Whitmore [17] and the ranking of geometric means condition derived by Jean [8] would be undisclosed by a mean-k-th order lower partial moment ranking.

Expansion of the mean-risk model to a mean-k-th order lower partial moment model may be futile for a more important reason. Just as the semi-variance ranking was not a necessary condition for fourth and higher degrees of stochastic dominance a k-th order lower partial moment ranking is not a necessary condition for k+2nd degree and higher degree stochastic dominance. With k=3, for example, the lower partial moment ranking would be

$M_{f3}(c) \geq M_{g3}(c)$

or $F_3(c) \leq G_3(c)$

This inequality could be violated, that is

$F_3(c) > G_3(c)$

could hold for some particular value of c, but the two distributions could still meet the inequalities for fifth degree stochastic dominance

$F_1(b) \leq G_1(b)$

$F_2(b) \leq G_2(b)$

$F_3(b) \leq G_3(b)$

$F_4(x) \leq G_4(x)$ for all x.

The mean-third order lower partial moment model could yield an investment ranking that would be reversed by a fifth degree stochastic dominance test when distributions with equal arithmetic means are compared. Since the fifth degree stochastic dominance test is based on decreasing risk aversion, it is easily seen that a mean-third order lower partial moment model can be inconsistent with the expected utility hypothesis. A security pricing model based on the k-th order lower partial moment such as that derived by Bawa and Lindenberg [2] could also be inconsistent with expected utility.

Mao [10] justified the mean-semivariance model on the basis of an investor utility function that was quadratic for x less than c and linear for x above c. In Fishburn's [3] consideration of a mean-k-th order lower partial moment model the utility function was also linear for x above c. A utility function with a linear segment does not have the characteristic of strictly decreasing risk aversion. There may be some special statistical distribution of security returns that allows accurate choices with the mean-semivariance model or the mean-k-th order lower partial moment model as the normal distribution does with the mean-variance model, but until that distribution is discovered the lower partial moment models should remain suspect.

REFERENCES

1. Vijay S. Bawa, "Optimal Rules for Ordering Uncertain Prospects," Journal of Financial Economics, 2 (1975), 95-121.
2. Vijay S. Bawa and Eric B. Lindenberg, "Capital Market Equilibrium in a Lower Partial Moment Framework," Journal of Financial Economics, 5 (1977), 189-200.
3. Peter C. Fishburn, "Mean-Risk Analysis with Risk Associated with Below-Target Returns," American Economic Review, 67 (1977), 116-126.
4. Josef Hadar and William R. Russell, "Rules for Ordering Uncertain Prospects," American Economic Review, 59 (1969), 25-34.
5. Giora Hanoch and Haim Levy, "The Efficiency Analysis of Choices Involving Risk," Review of Economic Studies, 36 (1969), 335-346.
6. William W. Hogan and James N. Warren, "Toward the Development of an Equilibrium Capital-Market Model Based on Semivariance," Journal of Financial and Quantitative Analysis, 9 (1974), 1-11.
7. William H. Jean, "Comparison of Moment and Stochastic Dominance Ranking Methods," Journal of Finance and Quantitative Analysis, 10 (1975), 151-161.
8. William H. Jean, "The Geometric Mean and and Stochastic Dominance," Journal of Finance, 35 (1980).
9. William H. Jean, "Risk Aversion Revisited," working paper, The University of Alabama, 1980.
10. James C. T. Mao, "Models of Capital Budgeting, E-V vs. E-S," Journal of Financial and Quantitative Analysis, 4 (1970), 657-675.

11. Harry M. Markowitz, Portfolio Selection: Efficient Diversification of Investments, Wiley, 1959.

12. Timothy J. Nantell and Barbara Price, "An Analytical Comparison of Variance and Semivariance Capital Market Theories," Journal of Financial and Quantitative Analysis, 14 (1979), 221-242.

13. James P. Quirk and Rubin Saposnik, "Admissibility and Measurable Utility Functions," Review of Economic Studies, 29 (1962), 140-146.

14. R. Burr Porter, "Semivariance and Stochastic Dominance: A Comparison," American Economic Review, 64 (1974), 200-204.

15. Robert C. Scott and Philip A. Horvath, "On the Direction of Preference for Moments of Higher Order Than the Variance," Journal of Finance, 35 (1980), 915-919.

16. Hassan Tehranian, "Empirical Studies in Portfolio Performance Using Higher Degrees of Stochastic Dominance," Journal of Finance, 35 (1980).

17. George A. Whitmore, "Third-Degree Stochastic Dominance," American Economic Review, 60 (1970), 457-459.