

UNIFORM PROGRESSIVITY,
RESIDUAL PROGRESSION AND
SINGLE-CROSSING

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ABSTRACT

The relationship between lower residual progression and "single-crossing" as necessary and sufficient conditions for one tax system to be uniformly more progressive than another (more progressive for all pre-tax income distributions) is clarified.

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1. Introduction.

An important problem in public finance is determining when one tax is more progressive than another. Dominance of the Lorenz curve for after-tax income is now widely accepted as the appropriate basis for considering one tax globally more progressive than another. The Lorenz dominance criterion has the advantage of providing a normative basis for (positive) progressivity comparisons. A disadvantage of this criterion is that progressivity comparisons are complicated by the dependence of the comparisons on the distribution of pre-tax income. Thus, it is desirable to find conditions under which global progressivity comparisons can be made independently of the pre-tax income distribution, that is, conditions under which one tax is uniformly more progressive than another.

In a widely cited paper, Jakobsson [1976] proposes lower residual progression as a necessary and sufficient condition for one tax to be uniformly more progressive than another. Specifically, Jakobsson states that one tax is uniformly more progressive than another if and only if (iff) the first tax has lower residual progression at all pre-tax incomes. Hemming and Keen [1984] (hereafter, HK) extend Jakobsson's work, and propose a "single-crossing" condition as necessary and sufficient for one tax to be uniformly more progressive than another.

HK's proof that the single-crossing condition is necessary for one tax to be uniformly more progressive utilizes Jakobsson's proof of the necessity of lower residual progression. Recently, Formby, Smith, and Sykes [1985] (hereafter, FSS) have shown, by constructing a counterexample, that lower residual progression is not a necessary condition for one tax to be uniformly more progressive than another. Thus, FSS's counterexample casts doubt on single-crossing as a necessary condition for uniform progressivity. The purpose of this paper is to consider exactly what HK have shown, and to demonstrate that single-crossing is in fact a necessary and sufficient condition for one tax to be uniformly more progressive than another.

2. Previous Results

Following Jakobsson and HK, the tax system T_1 is "more progressive" than the tax system T_2 iff the first after tax income distribution Lorenz dominates the second after-tax income distribution.¹ That is, T_1 is more progressive than T_2 iff

$$L_1(p) \geq L_2(p), \quad \forall p \in [0, 1], \quad (1)$$

where $L_i(p)$ is the Lorenz curve for after-tax income under tax system T_i . The $L_i(p)$ depend on the distribution of pre-tax income, denoted $G(y)$; Λ denotes the set of pre-tax income distributions. The tax system T_1 is "uniformly more progressive" (u.m.p.) than T_2 iff (1) holds for all pre-tax income distributions, i.e., for $\forall G \in \Lambda$. For any two functions h_1 and h_2 , h_1 "single-crosses" h_2 on the interval $Y = [\underline{y}, \bar{y}]$ iff there exists a $y^* \in Y$ such that

$$h_1(y) \geq h_2(y), \quad y \in [\underline{y}, y^*], \quad \text{and}$$

$$h_1(y) \leq h_2(y), \quad y \in [y^*, \bar{y}].$$

Let $n_1(y)$ denote net or after-tax income under tax system T_1 as a function of pre-tax income, and let $E_G(n_1(y))$ be mean after-tax income, given the pre-tax income distribution is G . Normalized net income is defined as $n_1^G(y) = n_1(y)/E_G(n_1(y))$; observe that $E_G(n_1^G(y)) = 1$ for $l = 1, 2$ and $\forall G \in \Lambda$. The residual progression of T_1 , or elasticity of $n_1(y)$ with respect to y , is denoted $\phi_1(y)$.

Jakobsson's and HK's work can be viewed as an analysis of the relationship between three conditions. These conditions are:

$$(R) \quad \phi_1(y) < \phi_2(y), \quad \forall y \in Y.$$

$$(S) \quad n_1^G(y) \text{ single-crosses } n_2^G(y), \quad \forall G \in \Lambda.$$

$$(T) \quad T_1 \text{ is u.m.p. than } T_2$$

In this notation, Jakobsson's Proposition 1 (p. 165) becomes

$$R = > T, \text{ and} \quad (2)a.$$

$$T = > R. \quad b.$$

HK's Proposition 3 (p. 378) becomes

$$S = > T, \text{ and} \quad (3)a.$$

$$T = >. \quad b.$$

It is clear from Jakobsson's and HK's proofs that (2)a. and (3)a. are correct. That is, both the residual progression condition, R , and the single-crossing condition, S , are sufficient for T_1 to be u.m.p. than T_2 .

The problem arises in finding necessary conditions for T_1 to be u.m.p. than T_2 . FSS have shown that the residual progression condition is not, in general, a necessary condition, i.e., (2)b. is not valid.

A careful reading of HK's proof of (3)b. reveals that they actually demonstrate that the residual progression condition implies the single-crossing condition, $R = > S$. For this to prove (3)b., $T = > S$, requires that $T = > R$. But FSS have shown that $T = > R$ does not hold. It follows that HK's proof is not sufficient to establish the single-crossing condition as necessary for T_1 to be u.m.p. than T_2 .

Further, HK show, by a counterexample, that $R = > S$ is not valid.² The effect of this is to establish the single-crossing condition, S , as a weaker sufficient condition for T than the residual progression condition, R . But FSS's counterexample implies that neither Jakobsson nor HK have demonstrated necessary conditions for one tax system to be u.m.p. than another.

3. An Alternative Proof.

This section demonstrates that the single-crossing condition, S , is necessary for T_1 to be u.m.p. than T_2 .³

It is helpful to collect some preliminary results. The normalization of net income implies that $E_G(n_1^G(y)) = E_G(n_2^G(y)) = 1$ for $\forall G \in \Lambda$. This in turn implies that the $n_1^G(y)$ must cross at least once, $\forall G \in \Lambda$. That is, for every $G \in \Lambda$ there exists a point $t_G \in Y$ such that $n_1^G(t_G) = n_2^G(t_G)$. Observe that multiple crossings have not yet been ruled out, so t_G is taken to be the smallest value at which $n_1^G(y) = n_2^G(y)$.

The after-tax Lorenz curve under T_1 is

$$L_1(p) = \int_{\underline{y}}^y n_1^G(u) dG(u), \quad 1 = 1, 2,$$

where $p = G(y)$. Now observe that

$$\begin{aligned} \int_{\underline{y}}^{\bar{y}} [n_1^G(u) - n_2^G(u)] dG(u) &= \int_{\underline{y}}^v [n_1^G(u) - n_2^G(u)] dG(u) \\ &\quad + \int_v^{\bar{y}} [n_1^G(u) - n_2^G(u)] dG(u), \end{aligned} \quad (4)$$

$\forall v \in Y$ and $\forall G \in \Lambda$. From the properties of the $n_1^G(y)$, the left hand side of (4) vanishes. Then (4) can be rewritten as

$$\int_{\underline{y}}^v [n_1^G(u) - n_2^G(u)] dG(u) = \int_v^{\bar{y}} [n_2^G(u) - n_1^G(u)] dG(u), \quad (5)$$

$\forall v \in Y$ and $\forall G \in \Lambda$. The left hand side of (5) is $L_1(p) - L_2(p)$, evaluated at $p = G(v)$. Thus,

$$L_1(G(v)) - L_2(G(v)) = \int_v^{\bar{y}} [n_2^G(u) - n_1^G(u)] dG(u), \quad (6)$$

$\forall v \in Y$ and $\forall G \in \Lambda$.

What must be shown is that T_1 u.m.p. than T_2 rules out multiple crossings for every pre-tax distribution.⁶ That is, suppose there is some distribution of pre-tax income, G_0 , for which there exists

$s_0 > t_0$ such that

$$\begin{aligned} n_1^0(y) &> n_2^0(y) \text{ for } y \in [y, t_0], \\ n_1^0(y) &< n_2^0(y) \text{ for } v \in [t_0, s_0], \text{ and} \\ n_1^0(y) &> n_2^0(y) \text{ for } y \in (s_0, \bar{y}]. \end{aligned}$$

It will be shown that the existence of G_0 leads to a contradiction.

The assumption that T_1 is u.m.p. than T_2 implies that (6) must be nonnegative for all v and all G . In particular, this must be true

when the pre-tax income distribution is G_0 and pre-tax income is equal to s_0 . For the right hand side of (6), evaluation at G_0 and s_0 yields

$$\int_{s_0}^{\bar{y}} [n^0_2(u) - n^0_1(u)] dG_0(u).$$

But, since $n^0_1(y) > n^0_2(y)$ for $y > s_0$, this expression must be negative, contradicting the assumption of Lorenz dominance. Thus, if T_1 is u.m.p. than T_2 , then the single-crossing condition, S, must hold. HK's Proposition 3 is correct.

4. Conclusion.

Combining the results obtained here with those of previous writers, the following relationships have been shown to hold. If one tax system has lower residual progression than another at all incomes, then the first tax system satisfies the single-crossing condition, which in turn implies that the first tax system is uniformly more progressive than the second. If one tax is uniformly more progressive than another, then it satisfies the single-crossing condition. On the other hand, it is not true that if one tax system is uniformly more progressive then it has lower residual progression at all incomes. Similarly, it is not true that if a tax system satisfies the single-crossing condition, then it has lower residual progression at all incomes.

NOTES

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1. The notation used is that of HK, except that tax systems are referenced by 1 and 2, instead of M and L. Also, the tax systems are assumed to be distinct.

2. The problem can be seen another way. Jakobsson's and HK's propositions are $R \Leftrightarrow T$, and $S \Leftrightarrow T$, which together imply $R \Leftrightarrow S$. But HK show that $\neg(R \Rightarrow S)$, which in turn implies $\neg(R \Leftrightarrow S)$. Thus, at least one of Jakobsson's and HK's propositions must be false.

3. The alternative proof given here is based on HK's proof of their Proposition 1 (p. 375).

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