dfsiglc.did/3-25-87
diski3

DISTRIBUTION-FREE STATISTICAL INFERENCE FOR GENERALIZED LORENZ CURVES

John A. Bishop, S. Chakraborti, and Paul D. Thistle

University of Alabama

March, 1987

Preliminary version: Not for quotation or publication. Comments appreciated.

DISTRIBUTION-FREE STATISTICAL INFERENCE FOR GENERALIZED LORENZ CURVES

I. Introduction

Lorenz curves, which show the cumulative proportion of income as a function of the cumulative proportion of the population, have been widely applied in the analysis of income inequality. Atkinson (1970), and others, have derived the relationship between Lorenz curves and social welfare.[1] For two income distributions with equal mean incomes, one income distribution Lorenz dominates the second if, and only if, the first income distribution is preferred by every social welfare function that puts a premium on greater equality.[2] The restriction to distributions with equal mean incomes severely limits the applicability of the Lorenz dominance criterion. In particular, characterization of efficiency-equity tradeoffs is difficult.

Shorrocks (1983) has recently proposed a technique for capturing efficieny-equity tradeoffs for income distributions when mean incomes are not equal. Shorrocks defines the generalized Lorenz curve as the curve "... constructed by scaling up the ordinary Lorenz curve by the mean of the distribution." (p. 6). The height of the generalized Lorenz curve indicates the level of incomes, while, as with the ordinary Lorenz curve, the convexity of

1. See, for example, Dasgupta, Sen, and Starrett (1973), and Rothschild and Stiglitz (1973).
2. Preference for equality is characterized as S-concavity of the social welfare function, and is equivalent to acceptance of Dalton's (1920) principle of transfers.

the generalized Lorenz curve reflects the dispersion of incomes. Thus, the generalized Lorenz curve takes account of both the level (efficiency) and the dispersion (inequality) in the distribution of income. One income distribution generalized Lorenz dominates another if the poorest, two poorest, three poorest, etc. individuals have greater total income under the first income distribution. Shorrocks shows that generalized Lorenz dominance is equivalent to preferrence by all increasing, anonymous, equality preferring social welfare functions. If the generalized Lorenz curves cross, then which distribution is "better" depends on the specific social welfare function.

This paper provides distribution-free statistical inference procedures for generalized Lorenz (GL) curves. The procedures are distribution-free in the sense that the asymptotic distributions of the test statistics do not depend on the distribution generating the data. In Section II we propose a test for comparing GL curves at a given population percentile (the point test) and a test for equality of two GL curves (the joint test). Multiple comparison and union-intersection tests are proposed in Section III. These inference procedures are closely related to the distribution-free inference procedures of Beach and Davidson (1983) and Beach and Richmond (1985) for ordinary Lorenz curves. Our tests reduce to Beach-Davidson's and Beach-Richmond's if the distributions have equal means. The results are illustrated with GL curves for U.S. household income for 1969 and 1979 in Section IV, and concluding remarks are offered in Section V..

II. A Joint Test for Generalized Lorenz Ordinates

Our objective is to derive statistical tests for the difference between two sets of GL ordinates. We assume that two independent microdata samples are available, and that both the measure of income and the recipient unit are chosen appropriately.[3]

Let Y denote income, and F denote the population distribution function for income. The distribution function is assumed to be twice continuously differentiable, strictly monotonic, and have a finite, positive mean, and finite variance. Consider a set of K fractions, p_i, such that $0 < p_1 < p_2 < \ldots < p_K = 1$. For a sample of N observations let r_i be the largest integer less than p_iN. Let ξ_i denote the p_ith population quantile, $F(\xi_i) = p_i$. The conditional mean of incomes less than ξ_i is $\gamma_i = E(Y|Y \leq \xi_i)$. The conditional variance of incomes less than ξ_i is $\lambda_i^2 = E[(Y - \gamma_i)^2|Y \leq \xi_i]$. Observe that γ_K and λ_K^2 are the overall or unconditional mean and variance.

The vector of GL ordinates at $p_1, \ldots, p_K$ is

$$\theta = (p_1\gamma_1,\ p_2\gamma_2,\ \ldots,\ p_K\gamma_K)' \qquad (1)$$

3. The choice of income measure and recipient unit is important for the social welfare interpretation of the GL curves. See Cowell (1984, pp. 357-363) for a discussion of this issue.

Observe that since $p_K = 1$, the K^{th} ordinate is the overall mean. The ordinary Lorenz curve ordinates, ϕ, are the GL ordinates, normalized by mean income:

$$\phi = \theta/\gamma_K = (p_1\gamma_1/\gamma_K, \ \ldots, \ p_{K-1}\gamma_{K-1}/\gamma_K, \ 1)'.$$

A Wald test for the difference between two vectors of GL ordinates, say, θ^a and θ^b can be constructed from a suitable function of the difference in the estimated ordinates, $\hat{\theta}^a - \hat{\theta}^b$. The choice of a suitable function is motivated by knowledge of the sampling distribution of the estimated GL ordinates. Beach and Davidson (1983) derive the aysmpototic distribution of the estimated GL ordinates.

Theorem (Beach and Davidson):
Under the assumptions above, $\sqrt{N}(\hat{\theta} - \theta)$ has an asymptotic K-variate normal distribution with mean zero and variance-covariance matrix Σ, where, for $i \le j$,

$$\sigma_{ij} = p_i[\lambda_i{}^2 + (1 - p_j)(\xi_i - \gamma_i)(\xi_j - \gamma_j) + (\xi_i - \gamma_i)(\gamma_j - \gamma_i)] \qquad (2)$$

All of the relevant quantities are consistently estimated by their sample analogs. For example, the quantile ξ_i is estimated by the $r_i{}^{th}$ sample order statistic, $y_{(ri)}$, and the conditional mean γ_i is estimated by the sample mean of the first r_i order statistics, $(1/r_i)\Sigma^{ri}y_{(j)}$. Beach-Davidson use this theorem to derive the asymptotic distributions of, and tests of hypothesis for, income shares and ordinary Lorenz curve ordinates. Beach-Richmond use this theorem to construct simultaneous confidence intervals for income shares and Lorenz ordinates.

Initially suppose we wish to test the equality of the ordinates at a given population precentile p_i. That is, we wish to test the null hypothesis $\theta^a{}_i = \theta^b{}_i$. It follows from Beach-Davidson's theorem that, under the null hypothesis, the statistic

$$Z_i = (\hat{\theta}^a{}_i - \hat{\theta}^b{}_i)/[\hat{\sigma}^a{}_{ii}/N^a + \hat{\sigma}^b{}_{ii}/N^b]^{1/2} \qquad (3)$$

has an asymptotic standard normal distribution.[4]

Now suppose we wish to test for the equality of two GL curves. That is, we wish to test the null hypothesis $\theta^a = \theta^b$. It also follows from Beach-Davidson's theorem that under the null hypothesis the difference in the vectors of estimated GL ordinates, $\hat{\theta}^a - \hat{\theta}^b$, has an asymptotic normal distribution with mean zero and variance-covariance matrix $\Sigma = \Sigma^a/N^a + \Sigma^b/N^b$. The test statistic for a joint test of equality of the GL ordinates is then

$$T = (\hat{\theta}^a - \hat{\theta}^b)'\Sigma^{-1}(\hat{\theta}^a - \hat{\theta}^b). \qquad (4)$$

Under the null hyopthesis, T is asymptotically distributed as χ^2 with K degrees of freedom.[5]

III. Simultaneous Inference for Generalized Lorenz Ordinates

To carry out the point test in (3) for multiple pairs of ordinates is a problem of simultaneous inference. In this section we develop a multiple comparison test and the

4. Observe that for the K^{th} pair of GL ordinates, this is the usual test for the difference between two means.
5. Gastwirth and Gail (1985) propose a test for the equality of ordinary Lorenz curve ordinate vectors based on the sum of the ordinates; the modification for GL curves is obvious. Gastwirth and Gail point out that their test has poor power against crossings.

implied joint test for equality of GL ordinates. Our inference procedures are based on Richmond's (1982) technique for constructing joint confidence intervals for all possible linear combinations of means of multivariate normal distributions. The advantage of Richmond's technique is that emphasis can be given to preselected linear combinations of the means that are of primary interest. Beach and Richmond (1985) apply this technique to construct joint confidence intervals for ordinary Lorenz curve ordinates and income shares. Richmond's technique uses the Studentized Maximum Modulus (SMM) distribution, which is relatively unfamiliar in econometrics. We now discuss the SMM distribution and the adaptation of Richmond's technique to hypothesis testing.

Let $x = (x_1, \ldots, x_K)' \sim N(\mu, \sigma^2 D)$, where D is a diagonal matrix, and let s^2 be an estimator of σ^2, where $ns^2/\sigma^2 \sim \chi^2(n)$ independently of x. Then the random variable $M = \max_i \{|x_i - \mu_i|/s\}$ has a Studentized Maximum Modulus distribution with K and n "degrees of freedom", abbreviated $M \sim SMM(K, n)$. Thus, $\Pr[M \geq m_\alpha(K, n)] = \alpha$, where $m_\alpha(K, n)$ is the upper $100(1-\alpha)\%$ point of the SMM(K, n) distribution. Hochberg (1974) has shown that if $x \sim N(\mu, \sigma^2 V)$, so the x_i are not independent, it is still true that $\Pr[M \geq m_\alpha(K, n)] \leq \alpha$; the equality holds if the x_i are independent. Since M is the largest of the $|x_i - \mu_i|/s$, it follows that

$$\Pr[|x_i - \mu_i|/s \geq m_\alpha(K, n), \; i = 1, \ldots, K] \leq \alpha.$$

Now consider a set of q known vectors, $\{b_1, \ldots, b_q\}$; for simplicity we assume $q \leq K$ and that the vectors b_j are linearly independent. The b_j are the "primary" vectors, and are chosen to select those linear combinations of means that are of primary interest. Define the $(q \times K)$ matrix B by $B' = [b_1 \ldots b_q]$. Then $Bx \sim N(B\mu, \sigma^2 BVB')$. Richmond shows that

$$\Pr[|b_i(x_i - \mu_i)|/s(b_i'Vb_i)^{1/2} \geq m_\alpha(q, n),\ i = 1, \ldots, q] \leq \alpha \tag{5}$$

Richmond further shows that the inequality in (5) holds for any arbitrary vector b in the linear space spanned by the primary vectors $\{b_1, \ldots, b_q\}$. Thus, the inequality in (5) yields joint $100(1\alpha)\%$ confidence intervals for all possible linear combinations of the means; the length of the confidence intervals is minimized for the set of linear combinations $\{b_1\mu, \ldots, b_q\mu\}$ of primary interest.

Beach and Richmond (1985) use this approach to derive joint confidence intervals for ordinary Lorenz curve ordinates and income shares. Similarly, this approach yields joint confidence intervals for GL ordinates. More importantly for our purposes, the inequality in (5) implicitly defines a multiple comparison test for pairs of GL ordinates. Define the vector $Z = (Z_1, \ldots, Z_K)'$, where the Z_i are the test statistics in (3); this vector is asymptotically normally distributed. Setting B equal to the K-dimensional identity matrix selects the GL ordinates as the primary points. The Hochberg-Richmond inequality (5) implies that $\Pr[|Z_i| \geq m_\alpha(K, \infty),\ i = 1, \ldots, K] \leq \alpha$. Thus, multiple comparisons of pairs of GL ordinates can be made on the basis of the statistics Z_i in (3), where the Z_i are tested as SMM(K, ∞) variates.[6]

The set of multiple comparison tests also implies a joint test for equality of the vectors of GL ordinates. The implied joint test accepts $H_0: \theta^a = \theta^b$ if, and only if, all of the component hypotheses H_k: $\theta^a{}_k = \theta^b{}_k$ are accepted. The rejection (acceptance) region of the implied joint test is the union (intersection) of the rejection (acceptance) regions of the individual tests. This implied joint test is a union-intersection test, or in Savin's (1980, 1984) terminology, a finite induced test. Observe that as a result of the inequality (5) the nominal size of the implied joint hypothesis test is α, the same as the size of the individual component tests.[7] The joint test in (4) can be interpreted as the infinite induced test where all possible linear combinations of the GL ordinates are equally important.

IV. Application to U.S. Income Distributions

6. Tables of the SMM distribution may be found in Stoline and Ury (1979). Our parameter K corresponds to Stoline and Ury's k^*. For a discussion of the SMM distribution and related multiple comparison techniques, see Miller (1981, pp. 12-30, 37-75).
7. Savin (1980, 1984) suggests applying the Bonferroni or Sidak inequalities to adjust the critical value of the component tests in order to maintain the nominal size of the induced joint test at α. If Z_c denotes the critical value, using the Bonferroni inequality implies $c = \alpha/2K$, and using the Sidak inequality implies $c = 1 - (1 - \alpha)^{1/K}$.

In this section we provide an empirical example to illustrate the application of the tests proposed in the previous section. We apply the tests to the GL curves for real U.S. household income in 1969 and 1979. These GL curves are shown in Figure 1. From the Figure, the GL curve for 1979 lies above the GL curve for 1969. The question is whether this is due to sampling varability or reflects a difference in the underlying income distributions.

The procedure for estimating the GL curves and carrying out the tests is as follows. First, two random samples of households were drawn from the 1970 and 1980 Census of Population and Housing. The samples contain 10,591 ~~16,997~~ households for 1969 and 11,696 households for 1979. The measure of income used is total household income, as reported by the Census. Since income is reported by the Census in current dollars, we adjusted for changes in the cost of living using the GNP implicit price deflator for personal consumption expenditures; all incomes are measured in 1979 dollars.[8] The households in each sample were then ranked separately from lowest to highest income. The sample conditional means for each decile were calculated, and the vector of estimated GL ordinates calculated according to (1).

The order statistics corresponding to the population deciles were then found; these estimate the quantiles ξ_i

8. Alternatively, one can estimate the GL ordinates and the variance-covariance matrices in nominal terms, then adjust for the change in the cost of living.

for i = 1, ..., 10. The sample condtional variances were calculated for each decile. The estimated quantiles, conditional means, and conditional variances were then used to estimate the variance-covariance matrix according to the formula in (2).

Table 1 reports the estimated GL ordinates for 1969 and 1979, along with the estimated standard errors. These are the GL ordinates plotted in Figure 1. The joint test statistic (3) has a calculated value of ~~169.50~~ 125.48. This greatly exceeds the 1% critical value of the x2(10) distribution, and we reject the hypothesis that the vectors of GL ordinates are equal. If we compare the ordinates at a given decile, the test statistic is Z in (3); the values of Z at each decile are reported in Table 1. For example, at the 10th decile, the test is for equality of the (overall) means; 1979 mean real income is significantly higher.

To test for equality of the ordinates at several deciles simultaneously, we use the multiple comparison test in (3) and (5). Operationally, this simply requires comparing the calculated Z_i in Table 1 to the critical value of the SMM distribution. The 1% critical value from the SMM(10, ∞) distribution is 3.289. This implies that equality of the vectors of GL ordinates is also rejected at the 1% level by the union-intersection test. We conclude that the difference in the GL curves shown in Figure 1 in not due to sampling variability, but reflects differences in the underlying distribution of income.

At each decile the GL ordinate for 1979 lies above the GL ordinate for 1969. Recall that the recipent unit is the household, and the measure of income is total annual household income. Suppose we are willing to accept these as the appropriate measures of the recipent unit and income. Suppose further that we a willing to accept the Pareto principle and the principle of transfers, so that our social welfare function is monotonic and S-concave. Then between 1969 and 1979 social welfare unambigously increased.

V. Conclusions.

This paper develops distribution-free statistical inference procedures for comparing generalized Lorenz curves. We develop several tests for equality of GL ordinates, including single, multiple comparson, and joint tests. Our procedures for GL curves extend the ordinary Lorenz curve results of Beach-Davidson and Beach-Richmond; our tests reduce to theirs if the means of the distributions are equal. The tests are valid under reasonable assumptions on the population income distribution functions, and the computations are straightforward.

When two income distributions have unequal means, the GL curves are the appropriate device for social welfare evaluation of the income distributions. The results presented here allow consensually valid statements regarding social welfare to be made from sample data on the basis of sound inferential procedures.

TABLE 1

Generalized Lorenz Ordinates, 1969 and 1979
U.S. Household Income[a]

Decile	Generalized Lorenz Ordinates 1969	1979	Z^b
.1	166.41 (5.78)	214.89 (5.84)	5.89
.2	633.66 (17.4)	750.29 (14.44)	5.74
.3	1428.05 (26.62)	1598.01 (24.32)	4.86
.4	2531.04 (35.35)	2754.47 (35.45)	4.32
.5	3963.52 (43.78)	4236.01 (47.23)	4.44
.6	5635.19 (51.98)	6056.80 (59.93)	5.04
.7	7636.19 (60.30)	8235.61 (72.41)	5.94
.8	10022.33 (70.19)	10847.84 (85.86)	6.93
.9	12986.51 (89.73)	14139.39 (105.02)	8.02
1.0	18053.89 (148.37)	19591.86 (146.67)	7.37

a. Income measured in 1979 dollars. Estimated standard errors in parentheses.
b. For tests on an individual pair of ordinates, the 1% critical value is 2.326. For multiple tests, the 1% critical value is 3.289 for the SMM(10, ∞) distribution.

REFERENCES

Atkinson, Anthony B., "On the Measurement of Inequality," *Journal of Economic Theory*, 2 (Sept. 1970), 244-263.

Beach, Charles M., and Russell Davison, "Distribution-Free Statistical Inference with Lorenz Curves and Income Shares," *Review of Economic Studies*, 50 (Oct. 1983), 723-735.

Beach, Charles M., and James Richmond, "Joint Confidence Intervals for Income Shares and Lorenz Curves," *International Economic Review*, 26 (June 1985), 439-450.

Cowell, Frank A., "The Structure of American Income Inequality," *Review of Income and Wealth*, 30 (Sept. 1984), 351-375.

Dalton, Hugh, "The Measurement of the Inequality of Incomes," *Economic Journal*, 30 (Sept. 1920), 348-361.

Dasgupta, Partha, Amartya Sen, and David Starrett, "Notes on the Measurement of Inequality," *Journal of Economic Theory*, 6 (April 1973), 180-187.

Hochberg, Y., "Some Generalizations of the T-method in Simultaneous Inference," Journal of Mutilvariate Analysis, 4 (1974), 224-234.

Miller, Rupert G., *Simultaneous Statistical Inference* (2e), New York, Springer, 1981.

Rothschild, Michael, and Joseph E. Stiglitz, "Some Further Results on the Measurement of Inequality," *Journal of Economic Theory*, 6 (April 1973), 188-204.

Richmond, James, "A General Method for Constructing Simultaneous Confidence Intervals," *Journal of the American Statistical Association*, 77 (June 1982), 455-460.

Savin, N.E., "The Bonferroni and Sheffe' Multiple Comparison Procedures," *Review of Economic Studies*, 47 (Jan. 1980), 255-273.

Savin, N.E., "Multiple Hypothesis Testing," in Zvi Griliches and Michael Intriligator (eds), *Handbook of Economietrics, Vol. II*, North-Holland, 1984.

Shorrocks, Anthony F., "Ranking Income Distributions," *Economica*, 50 (Feb. 1983), 3-17.

Stoline, Michael R., and Hans K. Ury, "Tables of the Studentized Maximum Modulus Distribution and an Application to Multiple Comparsons Among Means," *Technometrics*, 21 (Feb. 1979), 87-93.

U.S. Bureau of the Census,

DISTRIBUTION-FREE STATISTICAL INFERENCE FOR GENERALIZED LORENZ CURVES

John A. Bishop, S. Chakraborti,
and Paul D. Thistle

Working Paper No. 119

Working Paper Series
Department of Economics, Finance
and Legal Studies
College of Commerce and Business
Administration
University of Alabama

DISTRIBUTION-FREE STATISTICAL INFERENCE FOR GENERALIZED LORENZ CURVES

I. Introduction

Lorenz curves, which show the cumulative proportion of income as a function of the cumulative proportion of the population, have been widely applied in the analysis of income inequality. Atkinson (1970), and others, have derived the relationship between Lorenz curves and social welfare.[1] For two income distributions with equal mean incomes, one income distribution Lorenz dominates the second if, and only if, the first income distribution is preferred by every social welfare function that puts a premium on greater equality.[2] The restriction to distributions with equal mean incomes severely limits the applicability of the Lorenz dominance criterion. In particular, characterization of efficiency-equity tradeoffs is difficult.

Shorrocks (1983) has recently proposed a technique for capturing efficieny-equity tradeoffs for income distributions when mean incomes are not equal. Shorrocks defines the generalized Lorenz curve as the curve "... constructed by scaling up the ordinary Lorenz curve by the mean of the distribution." (p. 6). The height of the generalized Lorenz curve indicates the level of incomes, while, as with the ordinary Lorenz curve, the convexity of

1. See, for example, Dasgupta, Sen, and Starrett (1973), and Rothschild and Stiglitz (1973).
2. Preference for equality is characterized as S-concavity of the social welfare function, and is equivalent to acceptance of Dalton's (1920) principle of transfers.

the generalized Lorenz curve reflects the dispersion of incomes. Thus, the generalized Lorenz curve takes account of both the level (efficiency) and the dispersion (inequality) in the distribution of income. One income distribution generalized Lorenz dominates another if the poorest, two poorest, three poorest, etc. individuals have greater total income under the first income distribution. Shorrocks shows that generalized Lorenz dominance is equivalent to preferrence by all increasing, anonymous, equality preferring social welfare functions. If the generalized Lorenz curves cross, then which distribution is "better" depends on the specific social welfare function.

This paper provides distribution-free statistical inference procedures for generalized Lorenz (GL) curves. The procedures are distribution-free in the sense that the asymptotic distributions of the test statistics do not depend on the population distribution function. In Section II we propose a test for comparing GL curves at a given population percentile (the point test) and a test for equality of two GL curves (the joint test). Multiple comparison and union-intersection tests are proposed in Section III. These inference procedures are closely related to the distribution-free inference procedures of Beach and Davidson (1983) and Beach and Richmond (1985) for ordinary Lorenz curves. Our procedures are equivalent to Beach-Davidson's and Beach-Richmond's if the distributions have equal means. The results are illustrated with GL curves for

U.S. household income for 1969 and 1979 in Section IV, and concluding remarks are offered in Section V.

II. Classical Inference for Generalized Lorenz Ordinates

Our objective is to derive statistical tests for the equality of two sets of GL ordinates. We assume that two independent microdata samples are available, and that both the measure of income and the recipient unit are chosen appropriately.[3]

Let Y denote income, and F denote the population distribution function for income. The distribution function is assumed to be twice continuously differentiable, strictly monotonic, and have a finite, positive mean, and finite variance. Consider a set of K fractions, p_i, such that $0 < p_1 < p_2 < \ldots < p_K = 1$. For a sample of N observations let r_i be the largest integer less than p_iN. Let ξ_i denote the p_ith population quantile, $F(\xi_i) = p_i$. The conditional mean of incomes less than ξ_i is $\gamma_i = E(Y|Y \leq \xi_i)$. The conditional variance of incomes less than ξ_i is $\lambda_i^2 = E[(Y - \gamma_i)^2|Y \leq \xi_i]$. Observe that γ_K and λ_K^2 are the overall or unconditional mean and variance.

The vector of GL ordinates at $p_1, \ldots, p_K$ is

$$\theta = (p_1\gamma_1, p_2\gamma_2, \ldots, p_K\gamma_K)' \qquad (1)$$

3. The choice of income measure and recipient unit is important for the social welfare interpretation of the GL curves. See Cowell (1984, pp. 357-363) for a discussion of this issue.

Observe that since $p_K = 1$, the K^{th} ordinate is the overall mean. The ordinary Lorenz curve ordinates, ϕ, are the GL ordinates, normalized by mean income:

$$\phi = \theta/\gamma_K = (p_1\gamma_1/\gamma_K, \ldots, p_{K-1}\gamma_{K-1}/\gamma_K, 1)'.$$

A Wald test for the difference between two vectors of GL ordinates, say, θ^a and θ^b, can be constructed from a suitable function of the difference in the estimated ordinates, $\hat{\theta}^a - \hat{\theta}^b$. The choice of a suitable function is motivated by knowledge of the sampling distribution of the estimated GL ordinates. Beach and Davidson (1983) derive the aysmpototic distribution of the estimated GL ordinates.

Theorem (Beach and Davidson):
Under the assumptions above, $\sqrt{N}(\hat{\theta} - \theta)$ has an asymptotic K-variate normal distribution with mean zero and variance-covariance matrix Σ, where, for $i \le j$,

$$\sigma_{ij} = p_i[\lambda_i^2 + (1 - p_j)(\xi_i - \gamma_i)(\xi_j - \gamma_j) + (\xi_i - \gamma_i)(\gamma_j - \gamma_i)] \quad (2)$$

All of the relevant quantities are consistently estimated by their sample analogs. For example, the quantile ξ_i is estimated by the $r_i{}^{th}$ sample order statistic, $y_{(ri)}$, and the conditional mean γ_i is estimated by the sample mean of the first r_i order statistics, $(1/r_i)\Sigma^{r_i}y_{(j)}$. Beach-Davidson use this theorem to derive the asymptotic distributions of, and tests of hypothesis for, income shares and ordinary Lorenz curve ordinates. Beach-Richmond use this theorem to construct simultaneous confidence intervals for income shares and Lorenz ordinates.

Initially suppose we wish to test the equality of the GL ordinates at a given population precentile p_i. That is, we wish to test the null hypothesis $\theta^a{}_i = \theta^b{}_i$. It follows from Beach-Davidson's theorem that, under the null hypothesis, the statistic

$$Z_i = (\hat{\theta}^a{}_i - \hat{\theta}^b{}_i)/[\hat{\sigma}^a{}_{ii}/N^a + \hat{\sigma}^b{}_{ii}/N^b]^{½} \qquad (3)$$

has an asymptotic standard normal distribution.[4]

Now suppose we wish to test for the equality of two GL curves. That is, we wish to test the null hypothesis $\theta^a = \theta^b$. It also follows from Beach-Davidson's theorem that, under the null hypothesis, the difference in the vectors of estimated GL ordinates, $\hat{\theta}^a - \hat{\theta}^b$, has an asymptotic normal distribution with mean zero and variance-covariance matrix $\Omega = \Sigma^a/N^a + \Sigma^b/N^b$. The test statistic for a joint test of equality of the GL ordinates is then

$$T = (\hat{\theta}^a - \hat{\theta}^b)'\Omega^{-1}(\hat{\theta}^a - \hat{\theta}^b). \qquad (4)$$

Under the null hyopthesis, T is asymptotically distributed as χ^2 with K degrees of freedom.[5] To test for the equality of an estimated GL curve with a theoretical GL curve, simply replace $\hat{\theta}^b$ by the theoretical GL ordinates in (3) or (4).

III. Simultaneous Inference for Generalized Lorenz Ordinates

4. Observe that for the K^{th} pair of GL ordinates, this is the usual test for the difference between two means.
5. Gastwirth and Gail (1985) propose a test for the equality of ordinary Lorenz curve ordinate vectors based on the sum of the ordinates; the modification for GL curves is obvious. Gastwirth and Gail point out that their test has poor power against crossings.

To carry out the point test in (3) for multiple pairs of ordinates is a problem of simultaneous inference. In this section we develop a multiple comparison test and the implied joint test for equality of GL ordinates. Our inference procedures are based on Richmond's (1982) technique for constructing joint confidence intervals for all possible linear combinations of means of multivariate normal distributions. The advantage of Richmond's technique is that emphasis can be given to preselected linear combinations of the means that are of primary interest. Richmond's technique uses the Studentized Maximum Modulus (SMM) distribution, which is relatively unfamiliar in econometrics. We now discuss the SMM distribution and the adaptation of Richmond's technique to hypothesis testing.

Let $x = (x_1, \ldots, x_K)' \sim N(\mu, \sigma^2 D)$, where D is a diagonal matrix, and let s^2 be an estimator of σ^2, where $ns^2/\sigma^2 \sim \chi^2(n)$ independently of x. Then the random variable $M = \max_i\{|x_i - \mu_i|/s\}$ has a Studentized Maximum Modulus distribution with K and n "degrees of freedom", abbreviated $M \sim SMM(K, n)$. Thus, $\Pr[M \geq m_\alpha(K, n)] = \alpha$, where $m_\alpha(K, n)$ is the upper $100(1-\alpha)\%$ point of the SMM(K, n) distribution. Hochberg (1974) has shown that if $x \sim N(\mu, \sigma^2 V)$, so the x_i are not independent, it is still true that $\Pr[M \geq m_\alpha(K, n)] \leq \alpha$; the equality holds if the x_i are independent. Since M is the largest of the $|x_i - \mu_i|/s$, it follows that

$$\Pr[|x_i - \mu_i|/s \geq m_\alpha(K, n),\ i = 1, \ldots, K] \leq \alpha.$$

Now consider a set of q known vectors, $\{b_1, \ldots, b_q\}$; for simplicity we assume $q \le K$ and that the vectors b_i are linearly independent. The b_i are the "primary" vectors, and are chosen to select those linear combinations of means that are of primary interest. Define the (q x K) matrix B by $B' = [b_1 \ldots b_q]$. Then $Bx \sim N(B\mu, \sigma^2 BVB')$. Richmond shows that

$$\Pr[|b_i(x_i - \mu_i)|/s(b_i'Vb_i)^{1/2} \ge m_\alpha(q, n),\ i = 1, \ldots, q] \le \alpha \tag{5}$$

Richmond further shows that the inequality in (5) holds for any arbitrary vector b in the linear space spanned by the primary vectors $\{b_1, \ldots, b_q\}$. Thus, the inequality in (5) yields joint $100(1 - \alpha)\%$ confidence intervals for all possible linear combinations of the means; the length of the confidence intervals is minimized for the set of linear combinations $\{b_1\mu, \ldots, b_q\mu\}$ of primary interest. Beach-Richmond (1985) use this approach to derive joint confidence intervals for ordinary Lorenz curve ordinates and income shares. Similarly, this approach yields joint confidence intervals for GL ordinates.

More importantly for our purposes, the inequality in (5) implicitly defines a multiple comparison test for pairs of GL ordinates. Define the vector $Z = (Z_1, \ldots, Z_K)'$, where the Z_i are the test statistics in (3); this vector is asymptotically normally distributed. Setting B equal to the K-dimensional identity matrix selects the GL ordinates as the primary points. The Hochberg-Richmond inequality (5) implies that $\Pr[|Z_i| \geq m_\alpha(K, \infty), i = 1, \ldots, K] \leq \alpha$. Thus, multiple comparison tests of pairs of GL ordinates can be made on the basis of the statistics Z_i in (3), where the Z_i are tested as SMM(K, ∞) variates.[6]

The set of multiple comparison tests also implies a joint test for equality of the vectors of GL ordinates. The implied joint test accepts $H_0: \theta^a = \theta^b$ if, and only if, all of the component hypotheses H_K: $\theta^a{}_K = \theta^b{}_K$ are accepted. The rejection (acceptance) region of the implied joint test is the union (intersection) of the rejection (acceptance) regions of the individual tests. This implied joint test is a union-intersection test, or in Savin's (1980, 1984) terminology, a finite induced test. Observe that as a result of the inequality (5) the nominal size of the implied joint hypothesis test is α, the same as the size of the individual component tests.[7] The joint test in (4) can be interpreted as the infinite induced test where all possible linear combinations of the GL ordinates are equally important.

6. Tables of the SMM distribution may be found in Stoline and Ury (1979). Our parameter K corresponds to Stoline and Ury's K^*. For a discussion of the SMM distribution and related multiple comparison techniques, see Miller (1981, pp. 12-30, 37-75).
7. Savin (1980, 1984) suggests applying the Bonferroni or Sidak inequalities to adjust the critical value of the component tests in order to maintain the nominal size of the induced joint test at α. If Z_c denotes the critical value, using the Bonferroni inequality implies $c = \alpha/2K$, and using the Sidak inequality implies $c = 1 -(1 -\alpha)^{1/K}$.

IV. Application to U.S. Income Distributions

In this section we provide an empirical example to illustrate the application of the tests proposed in the previous sections. We apply the tests to the GL curves for real U.S. household income in 1969 and 1979. The 1969 and 1979 GL curves are shown in Figure 1. From the Figure, the GL curve for 1979 lies above the GL curve for 1969. The question is whether this is due to sampling varability or reflects a difference in the underlying income distributions.

The procedure for estimating the GL curves and carrying out the tests is as follows. First, two random samples of households were drawn from the 1970 and 1980 Census of Population and Housing. The samples contain 10,591 households for 1969 and 11,696 households for 1979. The measure of income used is total household income, as reported by the Census. Since income is reported by the Census in current dollars, we adjusted for changes in the cost of living using the GNP implicit price deflator for personal consumption expenditures; all incomes are measured in 1979 dollars.[8] The households in each sample were then ranked separately from lowest to highest income. The sample conditional means for each decile were calculated, and the vector of estimated GL ordinates calculated according to (1).

8. Alternatively, one can estimate the GL ordinates and the variance-covariance matrices in nominal terms, then adjust for the change in the cost of living.

The order statistics corresponding to the population deciles were then found; these estimate the quantiles ξ_i for i = 1, ..., 10. The sample condtional variances were calculated for each decile. The estimated quantiles, conditional means, and conditional variances were then used to estimate the variance-covariance matrix according to the formula in (2).

Table 1 reports the estimated GL ordinates for 1969 and 1979, along with the estimated standard errors. These are the GL ordinates plotted in Figure 1. The joint test statistic (4) has a calculated value of 125.48. This greatly exceeds the 1% critical value of the $\chi^2(10)$ distribution (= 23.2), and we reject the hypothesis that the vectors of GL ordinates are equal. If we compare the ordinates at a given decile, the test statistic is Z in (3); the values of Z at each decile are reported in Table 1. For example, at the 10th decile, the test is for equality of the (overall) means; 1979 mean real income is significantly higher.

To test for equality of the ordinates at several deciles simultaneously, we use the multiple comparison test in (3) and (5). Operationally, this simply requires comparing the calculated Z_i in Table 1 to the critical value of the SMM distribution. The 1% critical value from the SMM(10, ∞) distribution is 3.289. At each decile, equality of the GL ordinates is rejected at better than the 1% level. This implies that equality of the vectors of GL ordinates is also rejected at the 1% level by the union-intersection test. We conclude that the difference in the GL curves shown in Figure 1 in not due to sampling variability, but reflects differences in the underlying distribution of income.

At each decile the GL ordinate for 1979 lies above the GL ordinate for 1969. Recall that the recipient unit is the household, and the measure of income is total annual household income. Suppose we are willing to accept these as the appropriate measures of the recipient unit and income. Suppose further that we a willing to accept the Pareto principle and the principle of transfers, so that our social welfare function is monotonic and S-concave. Then between 1969 and 1979 there is an unambiguous increase in social welfare.

V. Conclusions.

This paper develops distribution-free statistical inference procedures for comparing generalized Lorenz curves. We develop several tests for equality of GL ordinates, including single, multiple comparson, and joint tests. Our procedures for GL curves extend the ordinary Lorenz curve results of Beach-Davidson and Beach-Richmond; our tests reduce to theirs if the means of the distributions are equal. The tests are valid under reasonable assumptions on the population income distribution functions, and the computations are straightforward.

When two income distributions have unequal means, the GL curves are the appropriate device for social welfare evaluation of the income distributions. The results presented here allow consensually valid statements regarding social welfare to be made from sample data on the basis of sound inferential procedures.

TABLE 1

Generalized Lorenz Ordinates, 1969 and 1979
U.S. Household Income[a]

Decile	Generalized Lorenz Ordinates 1969	1979	Z^b
.1	166.41	214.89	5.89
	(5.78)	(5.84)	
.2	633.66	750.29	5.74
	(17.4)	(14.44)	
.3	1428.05	1598.01	4.86
	(26.62)	(24.32)	
.4	2531.04	2754.47	4.32
	(35.35)	(35.45)	
.5	3963.52	4236.01	4.44
	(43.78)	(47.23)	
.6	5635.19	6056.80	5.04
	(51.98)	(59.93)	
.7	7636.19	8235.61	5.94
	(60.30)	(72.41)	
.8	10022.33	10847.84	6.93
	(70.19)	(85.86)	
.9	12986.51	14139.39	8.02
	(89.73)	(105.02)	
1.0	18053.89	19591.86	7.37
	(148.37)	(146.67)	

a. Income measured in 1979 dollars. Estimated standard errors in parentheses.
b. For tests on an individual pair of ordinates, the 1% critical value is 2.326. For multiple tests, the 1% critical value is 3.289 for the SMM(10, ∞) distribution.

REFERENCES

Atkinson, Anthony B., "On the Measurement of Inequality," *Journal of Economic Theory*, 2 (Sept. 1970), 244-263.

Beach, Charles M., and Russell Davison, "Distribution-Free Statistical Inference with Lorenz Curves and Income Shares," *Review of Economic Studies*, 50 (Oct. 1983), 723-735.

Beach, Charles M., and James Richmond, "Joint Confidence Intervals for Income Shares and Lorenz Curves," *International Economic Review*, 26 (June 1985), 439-450.

Cowell, Frank A., "The Structure of American Income Inequality," *Review of Income and Wealth*, 30 (Sept. 1984), 351-375.

Dalton, Hugh, "The Measurement of the Inequality of Incomes," *Economic Journal*, 30 (Sept. 1920), 348-361.

Dasgupta, Partha, Amartya Sen, and David Starrett, "Notes on the Measurement of Inequality," *Journal of Economic Theory*, 6 (April 1973), 180-187.

Hochberg, Y., "Some Generalizations of the T-method in Simultaneous Inference," Journal of Mutilvariate Analysis, 4 (1974), 224-234.

Miller, Rupert G., *Simultaneous Statistical Inference* (2e), New York, Springer, 1981.

Rothschild, Michael, and Joseph E. Stiglitz, "Some Further Results on the Measurement of Inequality," *Journal of Economic Theory*, 6 (April 1973), 188-204.

Richmond, James, "A General Method for Constructing Simultaneous Confidence Intervals," *Journal of the American Statistical Association*, 77 (June 1982), 455-460.

Savin, N.E., "The Bonferroni and Sheffe' Multiple Comparison Procedures," *Review of Economic Studies*, 47 (Jan. 1980), 255-273.

Savin, N.E., "Multiple Hypothesis Testing," in Zvi Griliches and Michael Intriligator (eds), *Handbook of Economietrics, Vol. II*, North-Holland, 1984.

Shorrocks, Anthony F., "Ranking Income Distributions," *Economica*, 50 (Feb. 1983), 3-17.

Stoline, Michael R., and Hans K. Ury, "Tables of the Studentized Maximum Modulus Distribution and an Application to Multiple Comparsons Among Means," *Technometrics*, 21 (Feb. 1979), 87-93.

U.S. Bureau of the Census,

Generalized Lorenz Curves
For the United States