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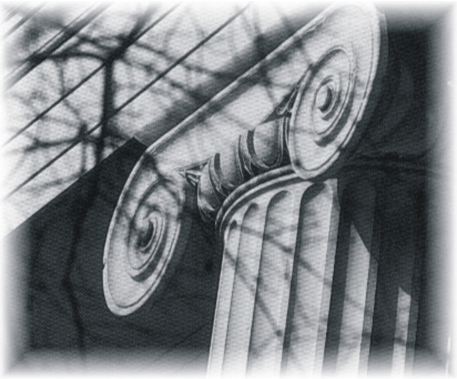

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Cournot Oligopoly and the Theory of Public Goods

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Abstract

We re-examine the model of Cournot oligopoly from the perspective of a public good provision problem. Utilizing this framework, we derive the Samuelson Marginal Condition which characterizes the set of collusive outcomes for the firms; show the Cournot/Nash equilibrium of the market is the outcome of firms voluntarily contributing to the provision of the public good; and use the public good nature of the problem to provide a natural recommendation for how collusive firms should divide up the market.

Keywords: Cournot Oligopoly, Public Goods, Ratio Equilibrium, Mechanism Design

1 Introduction

This paper applies the theory of public goods to the model of Cournot oligopoly.[1] While Cournot has certainly been well studied, a number of insights are made by formally redefining the model as a public good provision problem (treating the market price as a public good). This exercise contributes to the literature with a novel interpretation of the Cournot-Nash

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[1] See Cournot (1838). Excellent introductions to the large literature on Cournot can be found in Friedman (1983), Daugherty (1988), or Vives (1999).

equilibrium and provides a natural solution for colluding firms trying to optimally divide up market share.

The framework used in this paper has been around for a long time. In fact, the genesis for our ideas can be traced back to Mancur Olson's 1965 classic book *The Logic of Collective Action.* His book highlights many different examples, but the example of firm behavior in a perfectly competitive market exemplifies the type of public good problem analyzed in this paper.

> The firms in a perfectly competitive industry, for example, have a common interest in a higher price for the industry's product. Since a uniform price must prevail in such a market, a firm cannot expect a higher price for itself unless all other firms in the industry also have this higher price.[2]

He develops this analogy throughout the book to illustrate various public good issues (i.e., problems in provision arising from large numbers, division of cost, etc.), and also suggests that a similar exercise could be done using Cournot's model of oligopoly. Unfortunately, Olson's examples are largely informal and some subtleties in the public good analysis of oligopoly are lost as a consequence. We aim to formalize and extend several of Olson's insights about the Cournot model taking advantage of the literature that has developed since his canonical work.

Our analysis begins with the technical translation between the Cournot model and the public good provision model. This is simple. It follows from two facts about the oligopoly problem: first, as already mentioned, the market price *consumed* by the firms can be defined as a public good for the firms since it is both non-rival and non-excludable; second, firms indirectly "produce" this good via their choices of output through the demand function. In other words, the demand function takes the role of a production function in the public good provision model (with output reduction taken as the input). This formalism allows us to derive items such as the Samuelson Marginal Condition, a voluntary contribution mechanism equivalent to the Cournot quantity setting game, and the Lindahl/ Ratio equilibrium for the firm public good economy. These items each carry special interpretations within the Cournot model.

The Samuelson Marginal Condition characterizes the set of Pareto optimal allocations for a public good economy. In the Cournot model, we show

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[2] Olson (1965), p.9

that this condition is identical to the first order condition characterizing collusion.[3] This is not surprising, but the Samuelson Marginal Condition provides a useful marginal benefit/ marginal cost interpretation of the collusive first order condition. We use this condition as a baseline to conduct the rest of our analysis.

We next demonstrate that the standard Cournot game is equivalent to a game induced by a type of voluntary "contributions" mechanism. In this game, firms contribute quantities (or quantity reductions) to a planner who reduces (produces) the public good according to the demand function. Taking the decisions of their rivals as given, firms keep contributing until their "marginal rate of substitution" is equal to the price they pay for the public good (i.e., the real marginal cost). Since firms are paying the full cost of increasing the public good, they do not internalize the spillover benefits that accrue to the others and the public good is under provided –i.e., firms do not collude.[4]

Our main result uses the public good model to suggest a solution for collusive firms deciding on how to optimally split up the market. The standard theory is relatively silent in this respect. Some authors have proposed using axiomatic bargaining concepts such as the Nash Bargaining solution.[5] However, there is not any obvious motivation for using these concepts in the Cournot setting.[6] In contrast, we feel that the public good nature of the problem suggests using something akin to a Lindahl equilibrium to determine an acceptable collusive outcome. Olson also argues that the "right" way for the firms to share the cost of collusion is according to benefit.

> It would also seem that each individual in the group would bear a fraction of the total burden or cost, so that the burden of the of providing the public good would be shared in the "right"

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[3] The interior Samuelson Marginal Condition is where $\sum MRS_i = MC$ - i.e., the public good should be produced until the net social benefit is equal to the real cost of production. See Samuelson (1954).

[4] Since $\sum MRS > MC$.

[5] See, for example, Schmalensee (1987) or chapter 10 of Cross (1966) .

[6] Binmore (1987), for example, has argued that in some contexts, the information contained in messages should be taken into account when defining a bargaining solution. He shows, in a bilateral trade situation, that natural adaptations of Nash's Axioms to include information about the nature of the trades leads, not to the Nash Bargaining Solution, but to the *Walrasian Equilibrium.*

> way in the sense that the cost would be shared in the same proportion as the benefits.[7]

Unfortunately, Lindahl pricing does not provide a satisfying recommendation for the variety of environments we consider in this paper.

The modern definition of the Lindahl concept is due to Foley (1970). He shows that Lindahl allocations are in the core when the real marginal cost of producing the public good is constant. In our environment, this corresponds to linear market demand. However, when the real marginal cost of production is not constant, Lindahl equilibria need not be in the core or even exist.[8] Thus, if we desire a "core selecting" solution concept outside of markets with linear demand, the Lindahl equilibrium loses its attractiveness. This problem is remedied by appealing to the Ratio equilibrium concept due to Kaneko (1977).[9] This concept generalizes the Lindahl equilibrium, exists in a larger set of environments than Lindahl, and is always in the core. We define what it means to be a Ratio equilibrium for our firm economy, prove existence, and show, via an example, that the Ratio equilibrium outcome need not coincide with the Nash bargaining solution. We conclude the paper by illustrating how public good mechanisms can be used to non-cooperatively enforce our Ratio equilibrium concept using Corchón and Wilkie's 1996 Cost Share mechanism.

2 The Cournot Model as a Public Good Provision Problem

We consider a homogeneous product market with N firms, where each firm i cares about their consumption of a public good (the market price), denoted P, and a private good $q_i \in [0, \bar{q}]$ (their output). Firm i's preferences over a bundle (P, q_i) are represented by a utility function of the form

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[7] Olson (1965), p.27. Later in the chapter (pages 30-31) he is more specific and essentially suggests Lindahl pricing.

[8] Textbook examples of this can be found in Moore (2007), Chapter 16, Examples 16.12 and 16.18.

[9] See Kaneko (1977a,b) or Mas-Colell and Silvestre (1989). Mas-Colell and Silvestre present a concept called "Cost Share" equilibrium which is more general and contains the Ratio equilibrium as a special case. Other generalizations and axiomatizations appear in Diamantaras and Wilkie (1992) and van den Nouweland, Tijs, and Wooders (2002).

$\pi^i(P, q_i) = Pq_i - c_i q_i$, where $c_i \geq 0$ may vary between the firms. The rate at which a firm is willing to give up output (the private good) in order to get an additional unit of the price (the public good) is the marginal rate of substitution, defined, for each i, by $MRS_i = \frac{q_i}{P - c_i}$. Each private unit consumed is also converted into the public good according to the inverse demand/ production function $P = F(q_A + q_B)$, where, for $Q > 0$, F is a twice continuously differentiable, strictly decreasing, weakly concave, and $F(0) > \max_i c_i$.[10]

3 Pareto Efficiency in a Cournot Public Good Economy

We define the "efficient" allocations for the firms in this public good economy using 2 firms.[11] The Pareto problem is to maximize the profits of Firm A subject to resource constraints, production constraints, and the constraint that the profit of Firm B is at least as high as some benchmark m. Formally, we solve:

$$\begin{array}{llrcll} \max\limits_{P,\, q_A,\, q_B} \pi_A(P, q_A) & \text{s.t.} & P,\, q_A,\, q_B & \geq & 0 & \\ & & P & \leq & F(q_A + q_B) & : [\sigma] \\ & & \pi_B(P, q_B) & \geq & m & : [\lambda] \end{array}$$

where σ and λ are the corresponding Lagrangian multipliers. The Kuhn-Tucker conditions for P, q_A, and q_B are:

$[P]$:

$$\frac{\partial \pi_A}{\partial P} - \sigma + \lambda \frac{\partial \pi_B}{\partial P} \leq 0, \tag{1}$$

$[q_A]$:

$$\frac{\partial \pi_A}{\partial q_A} + \sigma \frac{\partial F}{\partial Q} \leq 0, \tag{2}$$

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[10] These conditions are sufficient to guarantee the existence of a Nash equilibrium. See Szidarovsky and Yakowitz (1977). More general existence conditions are found in Amir (1996).

[11] The N firm example is a simple extension. Also, there is nothing special about constant marginal cost for the derivation and subsequent proof.

$[q_B]$:

$$\lambda \frac{\partial \pi_B}{\partial q_B} + \sigma \frac{\partial F}{\partial Q} \leq 0, \tag{3}$$

where (1), (2), and (3) hold with strict equality if P, q_A, and q_B are positive respectively. Focusing on the interior conditions, putting (1), (2), and (3) together to eliminate the Lagrangian multipliers we arrive at the simplified expression

$$\frac{\frac{\partial \pi_A}{\partial P}}{\frac{\partial \pi_A}{\partial q_A}} + \frac{\frac{\partial \pi_B}{\partial P}}{\frac{\partial \pi_B}{\partial q_B}} = -\frac{1}{\frac{\partial F}{\partial Q}}.$$

This is just the Samuelson marginal condition (i.e.,$\sum MRS_i = MC$), where the *real* marginal cost of production is the reduction in output needed in order to get an additional unit increase in public good defined by $-\frac{1}{\frac{\partial F}{\partial Q}}$.

Proposition 1 *The set of collusive outcomes, without side payments, are characterized by the Samuelson Marginal Condition.*

Proof. The set of collusive outcomes are the ones that maximize π_A subject to $\pi_B \geq k$. The first order condition is $(F_1 q_A + F - c_A)(F_1 q_B)^{-1} = (F_1 q_A)(F_1 q_B + F - c_B)^{-1}$, where $F_1 = \frac{\partial F}{\partial Q}$. Divide both sides by $(F_1)^2 q_A\ q_B$ to get $1 + (F - c_A)(F_1 q_A)^{-1} = (1 + (F - c_B)(F_1 q_B)^{-1})^{-1}$. Now, multiply both sides by $1 + (F - c_B)(F_1 q_B)^{-1}$, expand the expression, and subtract 1 from both sides yielding $(F - c_B)(F_1 q_B)^{-1} + (F - c_A)(F_1 q_A)^{-1} + (F - c_B)(F - c_A)(F_1^2 q_B q_A)^{-1} = 0$. Finally, multiply both sides by $q_A q_B (F - c_B)^{-1}(F - c_A)^{-1}$, and take $\frac{1}{F_1}$ to the right hand side so that $\frac{q_A}{F - c_A} + \frac{q_B}{F - c_B} = -\frac{1}{F_1}$. This is the Samuelson Marginal Condition.[12] ■

4 Cournot as the Outcome of a Private Provision Mechanism

In the Cournot model, firms do not collude in a Nash equilibrium. This is usually shown by computing both the set of collusive outcomes and the set of

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[12] Note, that no part of the proof requires marginal cost to be constant. Indeed, for very general cost functions the Samuelson Marginal Condition characterizes the set of collusive outcomes.

Nash equilibria and remarking on the lack of any common points. However, explaining *why* firms do not collude is obscured by this type of analysis. A more instructive explanation is obtained by looking at our public good model. In particular, we gain insight into this problem by re-framing the Cournot model as a mechanism where firms voluntary contribute to the production of a public good.[13]

Definition 1: A *mechanism* specifies: (1) the choices or messages that each of the agents can send; and, (2), how those choices get mapped into an outcome.

The set of messages is called a message space, and the functions that take messages into allocations are called outcome functions.

In the Cournot Voluntary Contribution Mechanism (CVCM), firms simultaneously choose a message $q_i \in [0, \bar{q}]$ to contribute. Denote an arbitrary profile of N firm messages by $q \in [0, \bar{q}]^N$. The total contribution is pooled together and a public good is produced for the group according to the outcome function $P(q) = F(\sum_i q_i)$. The private good allocated to the firm is determined by the outcome function $q_i(q) = q_i$. Thus, for each message q the CVCM mechanism specifies an outcome $(P(q), q_1(q), ..., q_N(q))$. The firms interact with one another through their choices of the private good and the public good production function. Applying the firms' preferences over the outcomes determined by these messages we induce the familiar Cournot quantity setting game.

Consider Firm A's best response problem in a duopoly version of this game. For any contribution q_B, Firm A chooses contribution q_A to maximize profit given the outcome functions $F(\cdot)$ and $q_A(\cdot)$ – i.e., she solves $\max_{q_A} \pi_A(F(q_A + q_B), q_A)$. This is exactly the best reply problem in the standard Cournot analysis. Consequently, we get the same reaction equations and Nash equilibrium. However, it is still useful to proceed with the analysis. The Kuhn-Tucker marginal condition for the problem is $\frac{\partial \pi_A}{\partial P}\frac{\partial F}{\partial Q} + \frac{\partial \pi_A}{\partial q_A} \leq 0$, which holds with equality if the firm produces a positive output. In equilibrium, Firm A produces a positive amount. So our condition can be written as $\frac{\partial \pi_A}{\partial P} / \frac{\partial \pi_A}{\partial q_A} = -(\frac{\partial F}{\partial Q})^{-1}$ or $MRS_A = MC$. Adding up these conditions for both players yields $MRS_A + MRS_B > MC$. Thus, the public good is un-

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[13] This mechanism is similar in spirit to the private provision models used by Cornes and Sandler (1985), Bergstrom, Blume, and Varian (1986), or Androni (1988). The difference is the rate at which the private good is transformed into the public good.

der provided relative to the optimum. Since the firms do not receive all of the benefits for lowering their output, they reduce output too little relative to what they would do in a collusive arrangement. This is the prototypical public good provision story.

5 What is the "right" way for firms to collude?

Being illegal has not stopped economists from discussing what is the "best" way for firms to collude.[14] We continue this discussion by exploiting our public goods model to provide such a solution. Specifically, we look for a Ratio equilibrium of the firm economy. In this section, we define the Ratio equilibrium for the firms' public good provision problem, show that it always exists in our framework, and then work through several examples.

Definition 2: A *cost share system* is defined by a pair of ratios $r = (r_A, r_B)$ such that $r_i \geq 0$ and $r_A + r_B = 1$, where each individual i's cost share is given by

$$r_i c(P),$$

where $c(P) \equiv -F^{-1}(P)$ is the real cost of the public good (in terms of P).

A Ratio equilibrium is an allocation that, given a cost share system, maximizes each firm's profit subject to a "budget constraint."

Definition 3: A feasible allocation (P^*, q_A^*, q_B^*) is a *Ratio equilibrium*, given the ratio vector $r = (r_A, r_B)$, if and only if for each i:

$$q_i^* = w_i - r_i c(P^*) \text{ and } \pi^i(P, q_i) > \pi^i(P^*, q_i^*) \rightarrow q_i + r_i c(P) > w_i,$$

where $w_i \in \mathbb{R}$ is the value of i's initial endowment – i.e., if $(\mathring{P}, \mathring{q}_i)$ is i's initial endowment, then $w_i \equiv r_i c(\mathring{P}) + \mathring{q}_i$.

If a Ratio equilibrium exists, the equilibrium allocation is Pareto optimal. In equilibrium, each firm sets $MRS_i = r_i MC$. Adding up these conditions

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[14] Schmalensee (1987), for instance, looks at collusion where side payments are not possible and firms have different costs. He compares four axiom based solutions. Our appraoch is motivated by the public good nature of the problem.

gives us the Samuelson marginal condition for efficiency. Moreover, since each firm can always "demand" their Nash equilibrium allocation bundle, the Ratio allocation is individually rational. The Cournot environment has a Ratio equilibrium.

Proposition 2 *There exist a Ratio equilibrium of the N firm economy, where each firm i has preferences given by utility $\pi_i(P, q_i) = (P - c_i)q_i$ and owns the endowment (P^{NE}, q_i^{NE}).*

Proof. The proof consist of verifying that the firm economy satisfies the conditions of Kaneko's existence Theorem (See Kaneko (1977), p.127-129). Define the consumption set to be $[P^{NE}, \infty) \times \mathbb{R}_+$. Preferences for firm i are defined by $\pi_i(P, q_i) = (P - c_i)q_i$ and are easily verified to be continuos and increasing. Preferences are quasi-concave since the determinant of the bordered Hessian matrix is negative semi-definite. Next, define the cost of production to be $\tilde{C}(P) \equiv c(P) - c(P^{NE})$. Cost is increasing since $\frac{\partial \tilde{C}}{\partial P} = -(\frac{\partial F(F^{-1}(P))}{\partial P})^{-1} > 0$. The sign follows since demand is downward sloping. Cost is 0 at P^{NE} by construction. Finally, cost is convex since $\frac{\partial^2 \tilde{C}}{\partial P^2} = \left[\frac{\partial F(F^{-1}(P))}{\partial P}\right]^{-2} \frac{\partial^2 F(F^{-1}(P))}{\partial P^2} (\frac{\partial F(F^{-1}(P))}{\partial P})^{-1} > 0$. The sign follows from demand being downward sloping and concave. Last, $q_i^{NE} > 0$. Thus, the firm economy satisfies Kaneko's sufficient conditions for existence of a Ratio equilibrium. ■

We illustrate with two examples.

Example 1 Suppose the inverse demand for the market is linear of the form $P = \max\{0, 6 - q_1 - q_2\}$ and that Firm A and Firm B produce their outputs at unit costs $c_A = 1$ and $c_B = 2$ respectively. The unique Nash equilibrium of the Cournot quantity setting game $(q_A^{NE}, q_B^{NE}) = (2, 1)$. These choices by the firms correspond to a market price of $P^{NE} = 3$ and profits of 4 and 1 for Firms A and B respectively.

The Ratio equilibrium problem involves having each firm i maximize their utility function subject to a "budget constraint" taking as given the their respective cost ratio r_i. The problem for Firm A is $\max_{P, q_A} Pq_A - q_A - \lambda[r_A P + q_A - 3r_A - 2]$. This yields a demand of $P_A = 2 + \frac{1}{r_A}$. Similarly, Firm 2 solves $\max_{P, q_B} Pq_B - 2q_B - \lambda[r_B P + q_B - 3r_B - 1]$ for a demand of $P_B = \frac{5}{2} + \frac{1}{2r_B}$. Setting the two demands equal to one another and solving for for the r_A, $r_B \geq 0$ such that $r_A + r_B = 1$ we have $P_A = P_B$ only if $r_A = 2 - \sqrt{2} \approx 0.5857$ (which implies $r_B = 0.4143$). Thus, the level of the public good in a Ratio equilibrium is $P = 3.71$, produced by quantities $q_A = 1.59$ and $q_B = 0.7$.

Example 1 is an extension of a Nash Bargaining exercise found in Binmore's *Playing for Real* game theory text which uses the Nash Bargaining Solution as a collusive solution concept. The Nash Bargaining Solution, for this example, yields the payoffs $(\pi_A, \pi_B) = (4.31, 1.2)$, which are generated by outputs $(q_A^{NB}, q_B^{NB}) = (1.59, 0.70)$. This is exactly the same outcome that is produced by the Ratio equilibrium. However, the coincidence of solutions is not true in general as the next example illustrates.

Example 2 Suppose the inverse demand for the market is linear of the form $P = \max\{0, 6 - \frac{1}{3}q_A - \frac{1}{3}q_B\}$ and that Firm A and Firm B produce their outputs at unit costs $c_A = 1$ and $c_B = 3$ respectively. The unique Nash equilibrium of the Cournot quantity setting game is $(q_A^{NE}, q_B^{NE}) = (7, 1)$. The corresponding market price and profits are $P^{NE} = \frac{10}{3}$, $\pi_A^{NE} = \frac{49}{3}$, and $\pi_B^{NE} = \frac{1}{3}$. The Nash Bargaining Solution is found with numerical methods to be $(\pi_A, \pi_B) = (16.83, 0.47)$. This is generated by outputs $(q_A^{NB}, q_B^{NB}) = (6.08, 0.62)$. For the Ratio equilibrium, Firm A's demand is $P_A = \frac{13}{6} + \frac{7}{2r_A}$. Similarly, Firm B's demand is $P_B = \frac{19}{6} + \frac{1}{2r_B}$. Setting the two demands equal to one another and solving for equilibrium ratios gives us $r_1 = \frac{5}{2} - \frac{1}{2}\sqrt{11} \approx 0.8417$ and $r_2 = 0.1583$. The level of the public good in a Ratio equilibrium is $P = 6.33$, produced by quantities $q_1 = 4.48$ and $q_2 = 0.53$. The equilibrium profits are $\pi_1 = 23.87$, and $\pi_2 = 1.75$. Therefore the two solution concepts do not always yield the same result.

6 Supporting a Collusive Outcome

Finally, although the Ratio equilibrium presents firms with an efficient and individually rational way to collude, it is not a Nash equilibrium of the quantity setting game. In this section, we look at how the firms might support a Ratio equilibrium outcome as a non-cooperative equilibrium. One option would be argue that the firms are playing a repeated game and implement a collusive outcome through trigger strategies in the spirit of Friedman (1971).

This approach is fine. However, in keeping with the public good spirit of the paper, we next illustrate how a public good mechanism can Nash implement the Ratio outcome of our firm economy. In particular, we apply the Cost Share mechanism (CSM) due to Corchón and Wilkie (1996) which strongly Nash implements the Ratio correspondence.[15] Our presentation slightly augments their mechanism in order to handle a positive endowment of the public good and to allow for potentially decreasing the current level of public good production.[16]

Specifically, in the CSM, each firm sends a 2 dimensional message. For each firm i, let $M_i = [0,1] \times \mathbb{R}$ be i's message space, with generic element $m_i = (r_i, p_i)$. It is useful to think of r_i as a cost share proposal and p_i as a vote for the level of the public good. If we suppose $(\mathring{P}, \mathring{q}_1, ..., \mathring{q}_N) \in \mathbb{R}^{N+1}_{++}$ is the strictly positive initial endowment, where $\mathring{P} > \max_i\{c_i\}$, then the outcome functions for the mechanism are defined as follows:

$$
\begin{aligned}
P(m) &= \begin{cases} \sum_i p_i + \mathring{P} & \text{– if } \sum r_i = 1 \text{ and } \sum p_i \geq -\mathring{P} \\ \mathring{P} & \text{– otherwise} \end{cases} \\
q_i(m) &= \begin{cases} \mathring{q}_i - r_i[c(P(m)) - c(\mathring{P})] & \text{– if } \sum r_i = 1 \text{ and } \sum p_i \geq -\mathring{P} \\ \mathring{q}_i - \epsilon & \text{– otherwise.} \end{cases}
\end{aligned}
$$

In words, this mechanisms works as follows: The firms submit a message to a planner. The planner checks the messages to make sure that the real cost of production is *exactly* covered (i.e., $r_A + r_B = 1$) and that the final amount of the public good to be produced "makes sense" (i.e., $P(m) \geq 0$). If these conditions are met, then the public good is changed to meet demand and the cost of production is distributed between firms. If these conditions are not met, the mechanism reverts back to the initial endowment minus a small penalty ϵ (for example $\epsilon = \frac{1}{N}\min\{\mathring{q}_1, ..., \mathring{q}_N\}$ could be used). Applying each firm i's preferences to these outcomes we get $u_i(m) = \pi_i(P(m), q_i(m))$. Thus, the cost share mechanism induces a game.

The proof that this augmented mechanism induces a game whose Nash equilibrium outcomes coincide with the Ratio equilibria of our firm economy

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[15] Other mechanisms could also be used. See, for instance, Tian and Li (1994) or Tian (1994).

[16] This type of augmented Cost Share mechanism was used in Van Essen (2010) to implement Ratio outcomes in an anticommons setting.

closely follows Corchón and Wilkie (1996) and is therefore omitted.[17] Instead we illustrate with an example.

Example 3 Reconsider the economic environment from Example 1. The unique Nash equilibrium outcome was $(P^{NE}, q_A^{NE}, q_B^{NE}) = (3, 2, 1)$, and the unique Ratio equilibrium, using the Nash equilibrium outcome as the initial endowment, was $((r_A,\ r_B),\ P,\ q_A,\ q_B) = ((0.5857,0.4143,\ 3.71,\ 1.59,\ 0.7)$. First, we demonstrate that there is a Nash equilibrium whose allocation is the same as the above specified Ratio equilibrium. Consider the strategy profile $[(r_A,\ p_A),\ (r_B,\ p_B)] = [(0.5857,\ \frac{.71}{2}),\ (0.4143,\ \frac{.71}{2})]$ as a candidate equilibrium. This strategy profile yields the Ratio equilibrium outcome when applied to the mechanism. Since the Ratio equilibrium is individually rational, we know it is preferred to the initial endowment. Now, suppose Firm A chooses to optimally deviate to $(\hat{r}_A,\ \hat{p}_A)$. First, it must be the case that $\hat{r}_A = r_A$. Suppose not, then $\hat{r}_A + r_B \neq 1$ and the mechanism's outcome is $(3, 2-\epsilon)$ which worse for Firm A than the initial endowment. So, $\hat{r}_A = r_A$. Using the same reasoning, $\hat{p}_A + p_B \geq -3$.

Through his choice of p_A, Firm A can set the market price P to be anything. In other words, his best response is equivalent to solving

$$\max_P \pi_A(P,\ 2 - 0.5857[c(P) - c(3)])$$

However, we know $P = 3.71$ is the unique answer to this problem. Letting $P = \hat{p}_A + \frac{.71}{2} + 3$, Firm A's best response is $\hat{p}_A = \frac{.71}{2}$. Similar reasoning for Firm B shows his strategy is also a best response. We conclude the candidate strategy profile is a Nash equilibrium that yields the Ratio outcome.

Now suppose $[(r_A, p_A),\ (r_B, p_B)]$ is a Nash equilibrium. Then each firm is best responding to the other firm's strategy. It must be the case that both $r_A + r_B = 1$ and $p_A + p_B \geq -3$. If not, then the firms receive an allocation that is strictly less preferred then their initial endowment. One the firms, say A, could unilaterally deviate and achieve his initial allocation which is a contradiction. Since (r_A, p_A) is a best response to (r_B, p_B), $r_A + r_B = 1$, and each firm i's price must solve $\max_{p_i} \pi_i(p_A + p_B + 3, 2 - (1 - r_{-i})[c(p_A + p_B + 3) - c(3)])$. If we let $P = p_A + p_B + 3$ and recognize that each i takes $1 - r_{-i}$

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[17] Corchon and Wilkie actually prove that this mechanism strongly Nash implements the Ratio correspondence – i.e., there are no profitable coalitional deviations from equilibrium either.

is parametric, it is clear that a Nash equilibrium outcome must be a Ratio equilibrium outcome.

7 Conclusion

In this paper, we have shown the Cournot model can be translated into one of public good provision. We exploited this public good framework to make several insights into the oligopoly problem. Moreover, it is clear that this type of approach can be used to shed insight on a variety of other contexts outside of traditional public economics. Examples include, but are not limited to entry deterrence, decisions of board of directors, and budget cuts within an organization. Some of these examples have already been studied under a public good guise. For example, Gilbert and Vives (1986) look at a model entry deterrence with two incumbent firms choosing output as means to keep price low enough to deter a potential entrant. They remark that this problem is a binary public good provision problem and that firms actually provide too much entry deterrence. Van Essen (2010) uses the approach in this paper to analyze methods for improving efficiency in complementary monopoly problems. It should also be noted that further theoretical work is needed on this problem. For instance, we suggested in the paper that, given the public good aspect of the problem, the Ratio equilibrium provides a natural recommendation for how colluding firms should split up the market. It would be interesting to find out if one could perform an analysis akin to Binmore (1987) and see if our approach is justified axiomatically. However, this is outside the intended scope of this paper.

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