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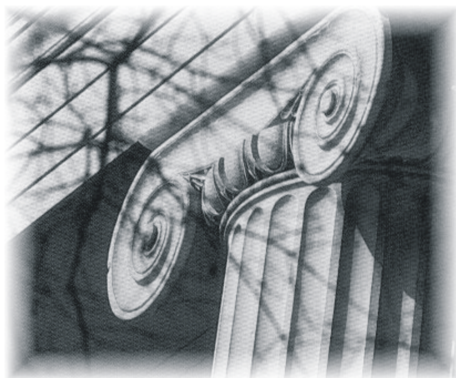

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Spread Options and Risk Management: Lognormal Versus Normal Distribution Assumption

Robert Brooks and Brandon N. Cline

Culverhouse College of Commerce and Business Administration
The University of Alabama

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Robert Brooks
Brandon N. Cline

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*The authors are Wallace D. Malone, Jr. Endowed Chair of Financial Management, University of Alabama; and Assistant Professor, Clemson University. Contact information: Robert Brooks, Department of Economics, Finance and Legal Studies, The University of Alabama, 200 Alston Hall, Box 870224, Tuscaloosa, AL 35487, rbrooks@cba.ua.edu, 205-348-8987.

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ABSTRACT

The difference between a spread option pricing model assuming both underlying instruments follow geometric Brownian motion, and a spread option pricing model assuming both underlying instruments follow arithmetic Brownian motion are often materially different from a risk management perspective. However, there are cases where the models are not significantly different. Material differences occur when time to maturity is high, spread volatility is high, or when spread correlation is low. This result is important in practical applications of risk management when spread options are included in the portfolio. There is a trade-off between the parsimonious, but internally inconsistent, normal distribution-based spread option model with potentially misleading risk parameters and the internally consistent lognormal distribution-based spread option model with accurate risk parameters. We also introduce a single integral solution to the bivariate lognormal case as well as introduce a new methodology to appraise risk management differences between models.

Spread Options and Risk Management: Lognormal Versus Normal Distribution Assumption

1. Introduction

Closed form solutions for valuing European spread options do not exist when the underlying instruments are assumed lognormally distributed; except for certain special cases such as exchange options (see Margrabe (1978)). When referring to "closed form", we use the finance definition of an expression whose numerical complexity is no greater than an "easy to compute" function of cumulative normal distribution functions. As a result, numerical techniques and approximations must be used for these types of spread option pricing models.

In this paper, we do not address the empirical question of which spread option pricing model is most suitable for particular spread options during particular periods of history. Instead we address a purely theoretical issue. Assuming both underlying instruments are known to be lognormally distributed, by erroneously assuming the spread is normally distributed, are various risk measures significantly biased? The examination of actual market data in an effort to identify the best model is not within the scope of this paper for a variety of reasons. First, it is very difficult to acquire clean market data as exchange-traded spread options are thinly traded and over-the-counter spread option data is not accessible.[1] Second, our focus here is on risk management and not spread option valuation. We assume the market option price is known and we seek to better understand the behavior of various risk measures. Finally, it is well known that the best valuation model for a particular product can change over time as market participants' perspectives change. Hence, even if we provided detailed empirical results it would not necessarily be applicable for other spread options or other periods of time.

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[1]A review of 16 papers related to spread option valuation revealed only one paper used actual spread option data. Alexander and Venkatramanan (2007) used less than one year of crack spread option data.

For options on underlying instruments other than spreads, current industry practice is to model the underlying instrument options assuming they are *lognormally* distributed. This assumption is often justified by the fact that the financial instrument prices are non-negative due to limited liability. Continuous time models based on geometric Brownian motion (GBM) imply the terminal distribution of the underlying instrument is lognormal. We refer to the spread option model which assumes both underlying instruments follow the lognormal distribution as the base model for comparison purposes. However, current industry practice is to model spread options by assuming the spread is *normally* distributed, primarily because the spread can be and often is negative. Continuous time models based on arithmetic Brownian motion (ABM) imply the terminal distribution of the underlying instrument is normal. We refer to spread option models using the normal distribution as the alternate models (to contrast them from the base model). These modeling procedures are internally inconsistent because the difference between two variables that are lognormally distributed does not follow a normal distribution.[2]

This internal inconsistency creates integrity concerns for risk systems. These integrity concerns are especially manifest for risk management of large portfolios. However, if it can be shown that the ABM assumption on the spread (alternate models) provides essentially the same results as the GBM (base models), the internal inconsistency can safely be disregarded for risk management purposes. Consequently, an alternate model using ABM can be deployed that has a much larger degree of tractability, thus minimizing the integrity concern.

This paper contributes to the spread option literature in three ways: First, we report a new simplified computational method for computing the value of spread options under GBM. Our option pricing model involves estimating three single integral expressions of the standard normal

[2]Poitras (1998) points out that since the difference of lognormal variables is not lognormal, a simplification of the Bachelier model is not possible.

cumulative distribution function (CDF). This procedure involves no iterative search routines or approximations outside of estimating a single integral and the standard normal CDF. Second, we provide a practical methodology for evaluating whether or not two option pricing models yield different risk parameters. Option pricing models are considered different if the measure of the difference in the risk parameters is larger than the measure of the difference in the risk parameters based on the bid-offer spreads. Finally, we illustrate model comparisons by varying moneyness, time to maturity, correlation, and the strike price. When the only concern is the risk parameter delta, we demonstrate that the ABM assumption does not cause a material error when the time to maturity is short or the correlation is high (spread volatility is low).

The paper will proceed as follows: Section 2 presents the two models under consideration: the lognormal spread option pricing model (LNSOPM) and the normal spread option pricing model (NSOPM). Section 3 provides the methodology for comparing two option models for risk management purposes (numerical examples used to illustrate the differences in the models from a risk management perspective are also included). Section 4 contains the conclusion.

2. Models

In this section, we provide two spread option valuation models, one assuming both underlying instruments follow GBM, the second assuming both underlying instruments follows ABM. The general framework of Black and Scholes (1973), Merton (1973) and Black (1976) is followed. Before delving into the particular spread option models, a brief review of the literature will be helpful.

Margrabe (1978) developed a closed-form equation for exchange options which are zero strike spread options. Poitras (1998) developed a pricing formula for European-style spread options by extending a special case of the Bachelier (1900) option-pricing model. The Bachelier model for pricing options on futures spreads provides a methodology for pricing European spread options, assuming changes in the underlying futures prices follow unrestricted arithmetic Brownian motion (UABM). The assumption of UABM proves very useful in that it allows negative sample paths to exist, resulting in call options that are priced higher than under plain arithmetic Brownian motion. Poitras points out that since the difference of lognormal variables are not lognormal, a simplification is not possible. Furthermore, he states that if prices are lognormally distributed, it is only possible to have a closed form solution in the special case of an exchange option. Otherwise, some double integral approximation must be used.

While it is true that the normality assumption should be questioned, Schaefer (2002) demonstrates that option values from the Bachelier model are nearly identical to values found using Monte Carlo simulation assuming GBM. However, this is only accurate provided the spread volatility and the time to maturity are both low.

Wilcox (1990) assumes ABM to derive a closed form solution for pricing spread options. However, it is shown by Poitras (1998) that the formula is not consistent with the no-arbitrage argument, thus it is not a valid option pricing formula. Despite the limitations of the Wilcox formula, many have used it to develop analytic approximations for spread option pricing including Shimko (1994).

Shimko (1994) provides an analytic approximation based on the Goldman/Wilcox model and a model developed by Rubinstein (1991). The Rubinstein model values options on futures spreads using a double-integral solution where both underlying futures contracts follow GBM. Like Rubinstein, Shimko assumes underlying prices follow GBM. However, by applying the Jarrow and Rudd (1982) approximations to the Wilcox model, Shimko approximates a single-integral solution for pricing options on futures spreads. At the same time, he assumes a stochastic convenience yield, thus overcoming the limitations of the Wilcox model and the complexity of the Rubinstein model. As a result, Shimko (1994) provides the formula to approximate the "true" lognormal solution.

Schaefer (2002), motivated by the fact that there were no closed form solutions for pricing options on futures spreads under the assumption that the assets follow correlated geometric Brownian motion, provides an analytic approximation for such options. Schaefer (2002) compares the Bachelier model to Monte Carlo simulation and the binomial methods that assume the spreads follow correlated geometric Brownian motion. For options on futures spreads with two or three underlying assets, his results indicate that both the Bachelier model and the analytic approximation provide solutions consistent with Monte Carlo simulation and binomial methods. However, he notes that as volatility and time to maturity increases the disparities between the models become much more pronounced, a result consistent with those reported here. Other analytic approximations have been published, including Alexander and Scourse (2004), Alexander and Venkatramanan (2007), Benth and Saltyte-Benth (2006), Minqiang, Deng and Zhou (2008), Carmona and Durrleman (2003, 2006), and Dempster and Hong (2000). Brooks (1995) provides a quadrinomial lattice approach to valuing spread options. Heenk, Kemna, and Vorst (1990) explore Asian options on oil spreads.

Pearson (1995) presents an efficient approximation to pricing spread options assuming the two underlying instruments are lognormally distributed. Pearson shows that the double integral can be reduced to a single integral (not counting the traditional N(d) integral). He proceeds to offer an efficient approximation to the resulting expression. The advantage of the lognormal model presented below is that it is not an approximation. Unfortunately, it does require integration of functions of the standard normal distribution. However, because very accurate approximations exist for the N(d) integral, standard single dimensional integration can be applied.

Borovkova, Permana, and Weide (2007) offer a spread option model based on a shifted lognormal distribution and then derive approximation formulas based on moments matching. Carmona and Durrleman (2003) provide a detailed overview of spread option pricing models and their uses as well as an approximation formula.

Let us review the two option pricing models for spread options before we turn to appraising model differences from a risk management perspective.

2.1. Lognormal Spread Option Pricing Model (LNSOPM)

We briefly review the assumptions and our notation for spread options. The general payoff at expiration, T, of a spread option can be expressed as:

$$CSO_T = \max\lfloor 0, \alpha_1 I_{1,T} + \alpha_2 I_{2,T} - X \rfloor$$
$$PSO_T = \max\lfloor 0, X - \alpha_1 I_{1,T} - \alpha_2 I_{2,T} \rfloor$$

where

CSO_T denotes the call option value at time T
PSO_T denotes put option value at time T
$\alpha_1 > 0$ denotes positive constant (index 1 coefficient)
$\alpha_2 < 0$ denotes negative constant (index 2 coefficient)
$-\infty < X < \infty$ denotes strike price
$I_{1,T} > 0$ denotes the value of index 1 at time T (stochastic)
$I_{2,T} > 0$ denotes the value of index 2 at time T (stochastic)

If we assume indexes follow geometric Brownian motion with geometric drift, then

$$dI_j = (\hat{\mu}_j - \delta_j) I_j dt + \hat{\sigma}_j I_j dz_j; j = 1,2$$

where

$-\infty < \hat{\mu}_j < \infty$ denotes the mean growth rate of index j

$-\infty < \delta_j < \infty$ denotes the carry costs related to index j

$\hat{\sigma}_j < \infty$ denotes the standard deviation of index j

$-\infty < dz_j < \infty$ denotes the standard Wiener process associated with index j

The value of spread option today can be expressed generically as,

$$SO_0 = PV\left[E_0\left(SO_T\right)\right]$$

where the expectation is taken under the equivalent martingale measure (standard finance assumptions are made – see Appendix A). The value of call and put options, using the *lognormal* distribution can be expressed as (base models):

$$\mathrm{CSO_1}\left(\mathrm{I_{1,0}, I_{2,0}, X, T}, \sigma_1, \sigma_2, \rho_{1,2}, \mathrm{r}\right) = \exp\{-\mathrm{rT}\}\left[\int_0^\infty\int_0^\infty \max\left[0, \alpha_1 \mathrm{I_{1,T}} + \alpha_2 \mathrm{I_{2,T}} - \mathrm{X}\right] \mathrm{f_1}\left(\mathrm{I_1, I_2}\right) \mathrm{dI_2 dI_1}\right]$$

$$\mathrm{PSO_1}\left(\mathrm{I_{1,0}, I_{2,0}, X, T}, \sigma_1, \sigma_2, \rho_{1,2}, \mathrm{r}\right) = \exp\{-\mathrm{rT}\}\left[\int_0^\infty\int_0^\infty \max\left[0, \mathrm{X} - \alpha_1 \mathrm{I_{1,T}} + \alpha_2 \mathrm{I_{2,T}}\right] \mathrm{f_1}\left(\mathrm{I_1, I_2}\right) \mathrm{dI_2 dI_1}\right]$$

where

1 = (subscript) denotes the LNSOPM,

r = risk-free interest rate; annualized with continuous compounding, and

$\mathrm{f_1}\left(\mathrm{I_1, I_2}\right)$ = bivariate lognormal density function.

Although a closed form solution to the LNSOPM does not exist, there are several single integral representations. Again, we assume the standard finance assumptions that afford using the risk-free rate as the mean for both indexes (see Appendix A). For example, the following single integral version of LNSOPM is used with a standard numerical integration methodology.[3]

$$\mathrm{CSO_{1,0}}\left(\mathrm{I_1, I_2}\right) = \mathrm{I_1 e^{-\delta_1 T}} \int_{-\infty}^{\infty} \mathrm{N}\left(\mathrm{d_{1,1}(z)}\right)\mathrm{n(z)dz} + \frac{\alpha_2 \mathrm{I_2}}{\alpha_1} \mathrm{e^{-\delta_2 T}} \int_{-\infty}^{\infty} \mathrm{N}\left(\mathrm{d_{1,2}(z)}\right)\mathrm{n(z)dz} - \frac{\mathrm{X}}{\alpha_1} \mathrm{e^{-rT}} \int_{-\infty}^{\infty} \mathrm{N}\left(\mathrm{d_2(z)}\right)\mathrm{n(z)dz}$$

$$\mathrm{PSO_{1,0}}\left(\mathrm{I_1, I_2}\right) = \mathrm{CSO_{1,0}}\left(\mathrm{I_1, I_2}\right) - \alpha_1 \mathrm{I_1 e^{-\delta_1 T}} - \alpha_2 \mathrm{I_2 e^{-\delta_2 T}} + \mathrm{Xe^{-rT}}$$

[3]Integral solving routines, such as Mathcad, can be used to find reduced form results such as this one. Although complex in appearance, N(d) is easily approximated and standard univariate integration routines can be used. Because bivariate integration is often unstable and therefore unreliable, this single integral solution is very useful.

where

$$n(z) = \frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}}$$

$$N(d_i(z)) = \int_{-\infty}^{d_i(z)} \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx$$

$$d_{1,1}(z) = \frac{\ln\left[\dfrac{\alpha_1 I_1 e^{(r-\delta_1)T + \rho^2 \frac{\sigma_1^2 T}{2} + \rho\sigma_1\sqrt{T}z}}{X - \alpha_2 I_2 e^{(r-\delta_2)T - \frac{\sigma_2^2 T}{2} + \rho\sigma_1\sigma_2 T + \sigma_2\sqrt{T}z}}\right] + (1-\rho^2)\dfrac{\sigma_1^2 T}{2}}{\sigma_1\sqrt{T(1-\rho^2)}}$$

$$d_{1,2}(z) = \frac{\ln\left[\dfrac{\alpha_1 I_1 e^{(r-\delta_1)T - \rho^2 \frac{\sigma_1^2 T}{2} + \rho\sigma_1\sigma_2 T + \rho\sigma_1\sqrt{T}z}}{X - \alpha_2 I_2 e^{(r-\delta_2)T - \frac{\sigma_2^2 T}{2} + \sigma_2^2 T + \sigma_2\sqrt{T}z}}\right] - (1-\rho^2)\dfrac{\sigma_1^2 T}{2}}{\sigma_1\sqrt{T(1-\rho^2)}}$$

$$d_2(z) = \frac{\ln\left[\dfrac{\alpha_1 I_1 e^{(r-\delta_1)T - \rho^2 \frac{\sigma_1^2 T}{2} + \rho\sigma_1\sqrt{T}z}}{X - \alpha_2 I_2 e^{(r-\delta_2)T - \frac{\sigma_2^2 T}{2} + \sigma_2\sqrt{T}z}}\right] - (1-\rho^2)\dfrac{\sigma_1^2 T}{2}}{\sigma_1\sqrt{T(1-\rho^2)}}$$

It is important to emphasize that this solution is not an approximation like Pearson (1995), Carmona and Durrleman (2003, 2006), Li, Deng and Zhou (2008) and others, rather it is an exact result. We do not, however, claim it is closed-form in the usual finance sense. It is still technically a double integral (recall the standard N(d) function is an integral). However, practically it is a single integral because of the existence of very accurate numerical approximations available to compute the standard N(d) function. Even the standard Black, Scholes, Merton option pricing model requires some sort of numerical approximation to N(d).

2.2. Normal Spread Option Pricing Model (NSOPM)

If we assume that the spread follows arithmetic Brownian motion with geometric drift, then

$$dS = \mu_S S dt + \sigma_S dz_S$$

$-\infty < \mu_S < \infty$ denotes the mean growth rate of the spread
$\sigma_S < \infty$ denotes the standard deviation of the spread (same units of measure as S)
$-\infty < dz_S < \infty$ denotes the standard Wiener process associated with the spread

The value of call and put spread options, based on the *normal* distribution can be expressed as (alternate model):

$$CSO_n\left(I_{1,0}, I_{2,0}, X, T, \sigma_1, \sigma_2, \rho_{1,2}, r\right) = \exp\{-rT\}\left[\int_0^\infty\int_0^\infty \max\left[0, \alpha_1 I_{1,T} + \alpha_2 I_{2,T} - X\right] f_n\left(I_1, I_2\right) dI_2 dI_1\right]$$

$$PSO_n\left(I_{1,0}, I_{2,0}, X, T, \sigma_1, \sigma_2, \rho_{1,2}, r\right) = \exp\{-rT\}\left[\int_0^\infty\int_0^\infty \max\left[0, X - \alpha_1 I_{1,T} + \alpha_2 I_{2,T}\right] f_n\left(I_1, I_2\right) dI_2 dI_1\right]$$

where

n = denotes the NSOPM, and

$f_n(I_1, I_2)$ = bivariate normal density function.

The advantage of the normal distribution is that the difference between normally distributed random variables is also normally distributed. Hence, there *does* exist a closed form solution to NSOPM.

Note that the spread is normally distributed and is denoted as:

$$S_T = \alpha_1 I_{1,T} + \alpha_2 I_{2,T}$$

Where the expected terminal spread is:

$$E[S_T] = \alpha_1 I_{1,0} e^{(\hat{\mu}_1 - \delta_1)T} + \alpha_2 I_{2,0} e^{(\hat{\mu}_2 - \delta_2)T} = S e^{\mu_s T}$$

and the variance of the spread is:

$$V[S_T] = \sigma_S^2 \frac{e^{2\mu_S T} - 1}{2\mu_S}$$

and σ_S is the standard deviation of changes in the spread. We assume the usual finance conditions that afford using the risk-free rate (see Appendix A). Therefore, we have the following version of NSOPM (alternate model):

$$CSO_{n,0}(I_1, I_2) = e^{-rT}\left[\{E[S_T] - X\}N(d_n) + V[S_T]^{1/2} n(d_n)\right]$$
$$PSO_{n,0}(I_1, I_2) = CSO_{n,0}(I_1, I_2) - \alpha_1 I_1 e^{-\delta_1 T} - \alpha_2 I_2 e^{-\delta_2 T} + Xe^{-rT}$$

where

$$n(d_n) = \frac{e^{\frac{-d_n^2}{2}}}{\sqrt{2\pi}}$$

$$N(d_n) = \int_{-\infty}^{d_n} \frac{e^{\frac{-d_n^2}{2}}}{\sqrt{2\pi}} dx$$

$$d_n = \frac{E[S_T] - X}{V[S_T]^{1/2}}$$

3. Analysis of Model Differences

As previously discussed, this paper does not seek to address a particular spread option, rather spread options in general. Therefore, rather than start with a set of option prices and calibrate both models, we generate option prices with the LNSOPM and calibrate the NSOPM so it generates the same original option prices. The advantage of this approach is that the NSOPM has only one spread option volatility parameter (under the normal distribution, volatility is in units not percent). When we compare appropriately calibrated option models we are not assuming one model is "correct." In practice, option prices are observable in the marketplace and an option pricing model is "calibrated" to the market option price. For example, one could compute the implied volatility. However, our objective is not to conduct an empirical test of a particular option contract. Our objective is solely to explore the implications for risk management by comparing the lognormal and normal models.

Therefore, we begin this analysis by calibrating the alternate model option price using the base model price. Calibration is a common practice for risk management applications. Traders take prices as given and then use models to infer risk parameters. We examine the resulting differences in risk parameters from the base GBM and the alternate ABM for spread options. By varying the parameters of the option models, we examine under what circumstances the risk parameters are significantly different.

Recall the objective here is to examine whether assuming the spread is normally distributed, when the two underlying instruments are lognormally distributed, results in materially different risk parameters. The LNSOPM requires three volatility parameters, the percentage standard deviation of each asset and the correlation between assets. The NSOPM only requires one volatility parameter, the per unit standard deviation of the spread. Hence, we assume the base model option prices (LNSOPM) are market prices and then calibrate the alternate model (NSOPM). For example, using the parameters specified later in Table 1, the spread volatility is \$20.8. However, the implied volatility from the NSOPM is \$18.9. The \$18.9 volatility generates the call price of \$7.56 as does the LNSOPM parameters.

If the distribution of the underlying instruments is normal, then the implied volatility of the spread option would match perfectly with the analytic spread volatility. However, the underlying instruments are assumed to follow a lognormal distribution and hence the difference of lognormal distributions is not lognormal. Thus, the implied volatility would not be expected to equal the analytic spread volatility. The spread option value computed with the LNSOPM will reflect the non-normal distribution. When the normal distribution is imposed, the NSOPM implied volatility will adjust the assumed normal distribution to reflect the LNSOPM price.

Once we have both the base and alternate models generating the same option prices, we estimate the risk parameters. It is important to emphasize that risk management is not directly an exercise in option valuation. Rather, option valuation models are used to estimate risk parameters, such as delta, gamma, theta, and vega. Finally, we assess the magnitude of the difference. By varying the initial inputs, we demonstrate that the pricing difference between the two models is often not materially different.

The question however arises: What is materially different? Specifically, from a risk management perspective what dictates a significant difference in risk parameters of the models being compared? Are deltas of 0.53 and 0.54 materially different? Our measure of materiality is based on the impact of trading costs on the value of risk parameters.

3.1. Significance Measure

The bid-ask spread represents an approximation of the marginal cost to the market maker for rebalancing the portfolio. There are clearly other costs, such as fixed costs, market impact costs, and other variable costs. To apply the methodology below with these other costs, one would have to appropriately adjust the estimated bid-ask spread. Our goal here is not to decompose the total spread option price change into the Greek components, rather merely to acknowledge that trading costs impact the ability to actually implement a hedging strategy. The higher the bid-ask spread, the less frequently the portfolio would be rebalanced, and therefore the risk parameters need not be as precise.

Our significance measure between the risk parameters of the base and alternate models therefore is a function of the bid-ask spread. Consider the following measure of significance.

Significance Measure: Assuming the market maker has the ability to rebalance the portfolio, an insignificant difference in risk parameter (RP) of the base and alternate models being compared exists if the risk parameter error factor between the models is less than the significance factor.

The significance factor of the base model is:

$$\hat{S} = \left| \frac{RP_{Base,Ask}(T) - RP_{Base,Bid}(T)}{\left[\frac{RP_{Base,Ask}(T) + RP_{Base,Bid}(T)}{2} \right]} \right|$$

and the risk parameter error factor between the base and alternate model is:

$$\varepsilon_{RP} = \left| \frac{RP_{Base}(T) - RP_{Alt}(T)}{\left[\frac{RP_{Base}(T) + RP_{Alt}(T)}{2} \right]} \right|$$

Thus, $\varepsilon_{RP} < \hat{S}$ implies $RP_{Base} \approx RP_{Alt}$.

Where,

$RP_{Base,Ask}$ = risk parameter of the option at the "ask" price, base model

$RP_{Base,Bid}$ = risk parameter of the option at the "bid" price, base model

RP_{Base} = risk parameter of base model

RP_{Alt} = risk parameter of alternate model

T = assumed transaction cost

$\approx$ = materially indistinguishable

The significance factor is a practical measure that seeks to incorporate how sensitive risk parameters are to the value of the underlying assets. If small deviations from the current market price of the underlying asset result in dramatic changes in the risk measures, then it will be difficult to hedge the position if trading costs are high. Therefore, the significance factor measures how sensitive the risk parameter is to changes in the underlying asset price due to the bid-offer spread. The error factor measures the deviation of the risk parameter between the two option models. Thus, an error factor that is lower than the significance factor would be deemed non-material, whereas an error factor in excess of the significance factor would be deemed material.

3.2. Illustrations

3.2.1. Moneyness

For the first illustration, we vary the moneyness of the option while holding all the other parameters fixed to examine the difference between the risk parameters of the base and alternate spread option pricing models. Our focus here is on risk management and not valuation. Hence, the observed market price is assumed to be based on the lognormal model. These market values are used to calibrate the alternate model so that they both yield the same price.

Moneyness of the option is varied from -30 to +30 units (for example, dollars), by varying the value of index 1. Time to maturity is fixed at one year, percentage volatility is 30 percent for each index, correlation is 0.8, strike price is 0.0, and the risk-free rate is 5 percent. A one percent trading cost is assumed on both indexes according to the method discussed in the previous section. By adjusting volatility, the option price of the NSOPM is calibrated to the price of the LNSOPM. The risk parameters are then calculated, and the error and significance factor are reported.

The sensitivity of option prices to changes in interest rates for the lognormal model is zero when the strike price is zero. Thus, Table 1 reports the results of the other four risk parameters: delta, gamma, theta, and vega.

We see that for the parameters selected the deltas are not materially different. That is, the error in delta between the two models is not greater than the difference in delta of the lognormal model, given a one percent transaction cost. Note that both the error factor and the significance factor are monotonically decreasing in moneyness for calls and monotonically increasing in moneyness for puts. Because call (put) deltas increase with moneyness, the percentage difference declines (increases).

The value of gamma is maximized around at-the-money spread options. Hence we observe a significance factor minimized when the options are at-the-money. However, gamma is materially different when the spread option is either in or out of the money. Very small changes in gamma are significant when the value of gamma is very small. Both in-the-money and out-of-the-money spread options result in small gammas and hence, the error factor is larger than the significance factor.

Both theta (option price sensitivity to time) and vega (option price sensitivity to volatility) are significant in all cases in Table 1. In both cases the significance factor is minimized at-the-money. Vega significance is the same for both calls and puts based on put-call parity for spread options. However, the risk parameter estimates are dramatically different between these two models. Time decay behavior is different between the lognormal and normal models, in part, due to its influence on volatility.

Volatility risk is difficult to manage, particularly for thinly traded options. We see here that the NSOPM measure of vega is materially different from the LNSOPM. Recall that the NSOPM has only one input for volatility, the volatility of the spread. However, the LNSOPM has three inputs, the volatilities of each underlying asset and the correlation. Therefore, there are several different measures of volatility risk that can be computed for the LNSOPM. We assume the volatility change was driven by a change in the volatility of the first asset. The goal here is to identify that volatility risk is an important consideration and can be assessed using the significance measure.

3.2.2. Time to maturity

Next we vary the time to maturity of the option while holding the other model parameters fixed to examine the difference between the risk parameters of the base and alternate spread option pricing models. Again, the market is assumed to provide current market values and these values are used to calibrate the alternate model so that both the base and alternate models yield the same price.

Time to maturity of the spread option is varied from 0.25 to 2 years. The remaining parameters are the same as in case 1. Index 1 and index 2 are assume to have values of 100.0 each and the strike price is 0, resulting in an at-the-money spread option.

Table 2 reports the results of this case. We observe that, for the parameters selected, the deltas are materially different for time horizons greater than one year. That is, the error in delta between the two models is not greater than the difference in delta of the lognormal model, given a one percent transaction cost, for maturities less than a year. Note that the error factor is monotonically increasing with time to maturity whereas the significance factor is monotonically decreasing with time to maturity. Hence, past some inflection point the models are materially different. For longer maturities, the two model deltas deviate further, whereas the impact of trading costs on deltas declines.

The significance factors for gamma in this case are very small because gammas are relatively large and trading costs do not influence its value much. For longer maturities, the gammas drift further apart. Hence, we conclude the models are materially different for longer maturities.

Once again, both theta and vega are significant in all cases in Table 2. The bid-offer spread does not influence these two risk parameters much but time to maturity does have a large impact on the difference between the models.

3.2.3. Correlation

We now vary the correlation of the option while holding the other parameters fixed to examine the difference between the risk parameters of the base and alternate spread option pricing models. Again, the market is assumed to provide current market values and these values are used to calibrate the alternate model so that both the base and alternate models yield the same price.

Correlation of the spread option is varied from -0.99 to +0.99. The remaining parameters are the same as in case 1. We assume a one year time to maturity and the rest of the parameters as previously reported. Varying the correlation is similar to varying the volatility since a spread option becomes more volatile as the correlation between the two indexes declines (holding the other parameters constant).

Table 3 reports the results of this case. We see that for the parameters selected the deltas are materially different when correlation declines (or volatility is increased). That is, the error in delta between the two models is greater than the difference in delta of the lognormal model, given a one percent transaction cost. Note that the error factor is monotonically decreasing with correlation and the significance factor is monotonically increasing. Hence, past some inflection point the models are no longer materially different. Therefore, as correlation declines (or volatility increases), the two models are materially different with respect to delta.

The significance factors for gamma in this case are quite unusual around a zero correlation as gamma oscillates around zero. This gamma effect results in unusual values for the significance factor. As the correlation increases, this becomes less of a problem and the significance factor declines.

Theta and vega results are consistent with the previous two tables.

3.2.4. Strike Price

We now vary the strike price of the option while holding the other parameters fixed to examine the difference between the risk parameters of the base and alternate spread option pricing models. Again, the market is assumed to provide current market values and these values are used to calibrate the alternate model so that both the base and alternate models yield the same price.

Varying the strike price is different than varying the index values as provided in Table 1 due to the influence of positive and negative strike prices. The strike price is varied from -30 to +30 by 10. The remaining parameters are the same as in case 1.

Table 4 reports the results of this case. We see that for the parameters selected the deltas are not materially different similar to Table 1. That is, the error in delta between the two models is greater than the difference in deltas of the lognormal model, given a one percent transaction cost. Note that the error factor is monotonically increasing with the strike price for calls and monotonically decreasing for puts.

The significance factors for gamma in this case are similar to Table 1. For high strike prices, the put theta is not significant. Similarly, for high strike prices, the call and put vegas are not significant.

4. Conclusion

Through four sets of option values, we demonstrate that the difference between two spread option pricing models, one assuming both underlying instruments follow GBM, and the other assuming both underlying instruments follow ABM, is not materially different in some cases, from a risk management perspective. As a result, the normal assumption on the underlying can be made, since it allows for an internally consistent model and provides a greater degree of tractability. However, for longer maturities and lower correlations (or higher volatilities), the two models are shown to be materially different. The particular model resulting in the most accurate risk parameters is an empirical issue and likely period and product specific.

Appendix A. Standard Finance Assumptions

Consider the standard set up for modeling prices (see, for example, Harrison and Kreps (1979) and Harrison and Pliska (1981):

1) $[0,\hat{\tau}]$, for fixed $\hat{\tau} \geq t \geq 0$, finite time horizon, a finite horizon economy, $0 \leq t \leq \hat{\tau}$.

2) $(\Omega, \Im, \mathrm{P})$, uncertainty is characterized by a complete probability space, where the state space Ω is the set of all possible realizations of the stochastic economy between time 0 and time $\hat{\tau}$ and has a typical element ω representing a sample path, $\Im$ is the sigma field of distinguishable events at time $\hat{\tau}$, and P is a probability measure defined on the elements of $\Im$.

3) $F = \{\Im(t): t \in [0,\hat{\tau}]\}$ the augmented, right continuous, complete filtration generated by the appropriate stochastic processes in the economy, and assume that $\Im(\hat{\tau}) = \Im$. The augmented filtration, $\Im(t)$, is generated by Z. $\Im(0)$ contains only Ω and the null sets of P.

4) F is generated by a K-dimensional Brownian motion, $Z(t) = [Z_1(t), \cdots, Z_K(t)]\, t \in [0,\hat{\tau}]$ is defined on $\{\Omega, \Im, P\}$, where $\{\Im(t)\}, t \in [0,\hat{\tau}]$ is the augmentation of the filtration $\{\Im^Z(t)\} t \in [0,\hat{\tau}]$ generated by $Z(t)$, and satisfies the usual conditions.

5) $E_{\mathrm{P}}(\cdot)$ denotes the expectation with respect to the probability measure P.

6) All stated equalities or inequalities involving random variables hold P-almost surely.

7) P is common for all agents implying uniqueness of the nature of the stochastic processes.

8) Conventional perfect market conditions are also assumed, such as no transaction costs, no taxes, unrestricted short selling, and no regulatory or institutional constraints.

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Table 1. Varying Moneyness
Panel A. Analysis of Delta

	Call Delta		Put Delta	
Moneyness	**Error Factor**	**Significance Factor**	**Error Factor**	**Significance Factor**
-30	0.2166	0.2340	0.0075	0.0080
-20	0.1571	0.1716	0.0234	0.0254
-10	0.1097	0.1212	0.0483	0.0530
0	0.0728	0.0811	0.0785	0.0868
10	0.0455	0.0514	0.1108	0.1239
20	0.0265	0.0302	0.1426	0.1605
30	0.0143	0.0166	0.1723	0.1970

Panel B. Analysis of Gamma and Theta

	Call and Put Gamma		Call and Put Theta	
Moneyness	**Error Factor**	**Significance Factor**	**Error Factor**	**Significance Factor**
-30	0.3591*	0.1832	0.6751*	0.2028
-20	0.2202*	0.1134	0.6682*	0.1296
-10	0.1077*	0.0427	0.6684*	0.0644
0	0.0098	0.0134	0.6690*	0.0056
10	0.1536	0.1566	0.6685*	0.0476
20	0.1976*	0.1486	0.6678*	0.0959
30	0.2102*	0.0960	0.6688*	0.1402

Panel C. Analysis of Vega

	Call Vega		Put Vega	
Moneyness	**Error Factor**	**Significance Factor**	**Error Factor**	**Significance Factor**
-30	0.5555*	0.2158	0.2932*	0.2158
-20	0.4396*	0.1386	0.2238*	0.1386
-10	0.3715*	0.0692	0.1847*	0.0692
0	0.2562*	0.0561	0.1231*	0.0561
10	0.3712*	0.0512	0.1825*	0.0512
20	0.4196*	0.1037	0.2076*	0.1037
30	0.4842*	0.1507	0.2422*	0.1507

* denotes the error factor being greater than the significance factor.

Table 1: This table reports the error and significance factor for selected risk parameters for the LNSOPM and NSOPM when the option prices are calibrated together. Moneyness of the option is varied from -30 to 30. Time to maturity is set equal to one year, volatility is 30 percent for both indexes, the risk-free rate is 5 percent, no dividends are assumed, and the correlation coefficient is 0.8. Index 2 is set to 100.0 and the strike price is set to zero. Index 1 is varied in increments of 10.0, allowing moneyness to vary from –30.0 to +30.0. Trading costs are assumed to be one percent for both assets; hence, buying the spread would result in a one percent decline in index 1 and a one percent increase in index 2.

Table 2. Varying Time to Maturity: Panel A. Analysis of Delta

	Call Delta		Put Delta	
Time to Maturity	**Error Factor**	**Significance Factor**	**Error Factor**	**Significance Factor**
0.25	0.0372	0.1758	0.0386	0.1604
0.50	0.0521	0.1196	0.0550	0.1182
0.75	0.0635	0.0954	0.0678	0.0987
1.00	0.0728	0.0811	0.0785	0.0868
1.25	0.0810*	0.0717	0.0881*	0.0787
1.50	0.0884*	0.0647	0.0970*	0.0727
1.75	0.0951*	0.0592	0.1051*	0.0679
2.00	0.1013*	0.0549	0.1128*	0.0641

Panel B. Analysis of Gamma and Theta

	Call and Put Gamma		Call and Put Theta	
Time to Maturity	**Error Factor**	**Significance Factor**	**Error Factor**	**Significance Factor**
0.25	0.0020	0.0057	0.6483*	0.0243
0.50	0.0126	0.0257	0.6637*	0.0114
0.75	0.0009	0.0059	0.6671*	0.0075
1.00	0.0098	0.0134	0.6690*	0.0056
1.25	0.0115*	0.0109	0.6704*	0.0045
1.50	0.0096*	0.0021	0.6718*	0.0037
1.75	0.0159*	0.0039	0.6732*	0.0032
2.00	0.0118*	0.0011	0.6747*	0.0028

Panel C. Analysis of Vega

	Call Vega		Put Vega	
Time to Maturity	**Error Factor**	**Significance Factor**	**Error Factor**	**Significance Factor**
0.25	25.5117*	0.0353	1.1799*	0.0353
0.50	2.4064*	0.0371	0.6109*	0.0371
0.75	0.0503*	0.0450	0.1940*	0.0450
1.00	0.2562*	0.0560	0.1231*	0.0560
1.25	0.6729*	0.0675	0.3721*	0.0675
1.50	0.0378*	0.0792	0.5721*	0.0792
1.75	1.1217*	0.0910	0.7361*	0.0910
2.00	1.2566*	0.1031	0.8724*	0.1031

* denotes the error factor being greater than the significance factor.

Table 2: This table reports the error and significance factor for selected risk parameters for the LNSOPM and NSOPM when the option prices are calibrated together. Volatility is 30 percent for both indexes, the risk-free rate is 5 percent, no dividends are assumed, and the correlation coefficient is 0.8. Index 1 and index 2 are set to 100.0 and the strike price is set to zero. Time to maturity is varied in increments 0.25, allowing time to maturity to vary from 0.25 to 2.00. Trading costs are assumed to be one percent for both assets; hence, buying the spread would result in a one percent decline in index 1 and a one percent increase in index 2.

Table 3. Varying Correlation: Panel A. Analysis of Correlation

	Call Delta		Put Delta	
Correlation	Error Factor	Significance Factor	Error Factor	Significance Factor
-0.99	0.2105*	0.0215	0.2666*	0.0337
-0.75	0.1990*	0.0227	0.2484*	0.0346
-0.50	0.1860*	0.0251	0.2284*	0.0369
-0.25	0.1714*	0.0281	0.2068*	0.0397
0.00	0.1550*	0.0320	0.1834*	0.0432
0.25	0.1358*	0.0381	0.1572*	0.0490
0.50	0.1125*	0.0483	0.1268*	0.0582
0.75	0.0811*	0.0717	0.0882	0.0788
0.99	0.0168	0.4365	0.0171	0.3127

Panel B. Analysis of Gamma and Theta

	Call and Put Gamma		Call and Put Theta	
Correlation	Error Factor	Significance Factor	Error Factor	Significance Factor
-0.99	0.0406*	0.0103	0.7006*	0.0240
-0.75	0.0446*	0.0360	0.6895*	0.0007
-0.50	0.0287*	0.0170	0.6862*	0.0007
-0.25	0.0675*	0.1007	0.6829*	0.0009
0.00	0.0300	0.0189	0.6798*	0.0011
0.25	0.0225*	0.0186	0.6762*	0.0015
0.50	0.0150*	0.0178	0.6729*	0.0023
0.75	0.0075*	0.0156	0.6696*	0.0045
0.99	0.0007*	0.0910	0.6665*	0.1116

Panel C. Analysis of Vega

	Call Vega		Put Vega	
Correlation	Error Factor	Significance Factor	Error Factor	Significance Factor
-0.99	0.1335*	0.0504	0.0489	0.0504
-0.75	0.1574*	0.0506	0.0592*	0.0506
-0.50	0.1794*	0.0509	0.0695*	0.0509
-0.25	0.1996*	0.0510	0.0798*	0.0510
0.00	0.2176*	0.0513	0.0901*	0.0513
0.25	0.2318*	0.0516	0.1005*	0.0516
0.50	0.2464*	0.0525	0.1107*	0.0525
0.75	0.2550*	0.0548	0.1210*	0.0548
0.99	0.2509*	0.1697	0.1308*	0.1697

* denotes the error factor being greater than the significance factor.

Table 3: This table reports the error and significance factor for selected risk parameters for the LNSOPM and NSOPM when the option prices are calibrated together. The initial parameters are the same as the previous tables. Correlation is varied in increments of 0.25, allowing correlation to vary from -0.99 to 0.99.

Table 4. Varying Strike Price: Panel A. Analysis of Delta

	Call Delta		Put Delta	
Strike Price	**Error Factor**	**Significance Factor**	**Error Factor**	**Significance Factor**
-30	0.0133	0.0156	0.1626	0.1879
-20	0.0259	0.0291	0.1401	0.1559
-10	0.0462	0.0511	0.1114	0.1221
0	0.0728	0.0811	0.0785	0.0868
10	0.1027	0.1157	0.0498	0.0556
20	0.1339	0.1491	0.0292	0.0322
30	0.1666	0.1787	0.0163	0.0174

Panel B. Analysis of Gamma and Theta

	Call and Put Gamma		Call Theta		Put Theta	
Strike Price	**Error Factor**	**Significance Factor**	**Error Factor**	**Significance Factor**	**Error Factor**	**Significance Factor**
-30	0.4108*	0.2653	1.5844*	0.7422	0.6490*	0.1384
-20	0.1374*	0.0010	0.8562*	0.1491	0.6441*	0.0973
-10	0.1213	0.1257	0.7160*	0.0573	0.6505*	0.0497
0	0.0098	0.0134	0.6690*	0.0056	0.6690*	0.0056
10	0.0744*	0.0260	0.6506*	0.0604	0.7160*	0.0703
20	0.1753*	0.0813	0.6441*	0.1075	0.8562*	0.1748
30	0.2582*	0.1290	0.6446*	0.1465	1.5683	2.1787

Panel C. Analysis of Vega

	Call Vega		Put Vega	
Strike Price	**Error Factor**	**Significance Factor**	**Error Factor**	**Significance Factor**
-30	0.0817	0.0848	0.0411	0.0848
-20	0.1695*	0.0440	0.0824*	0.0440
-10	0.2312*	0.0041	0.1116*	0.0041
0	0.2562*	0.0561	0.1231*	0.0561
10	0.2351*	0.1078	0.1117*	0.1078
20	0.1756*	0.1542	0.0819	0.1542
30	0.0921	0.1934	0.0419	0.1935

* denotes the error factor being greater than the significance factor.

Table 4: This table reports the error and significance factor for selected risk parameters for the LNSOPM and NSOPM when the option prices are calibrated together. The strike price is varied from -30 to +30 in increments of 10. Volatility is 30 percent for both indexes, the risk-free rate is 5 percent, no dividends are assumed, time to maturity is 1 year, and the correlation coefficient is 0.8. Index 1 and index 2 are set to 100.0 and the strike price is set to zero. Trading costs are assumed to be one percent for both assets; hence, buying the spread would result in a one percent decline in index 1 and a one percent increase in index 2.